Solución parcial 1

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I. Polinomios de Chebyshev I

1) En la ecuación

$$(1-x^2)\,P_n^{(\alpha,\beta)\,''} - (\alpha-\beta+(2+\alpha+\beta)x)\,P_n^{(\alpha,\beta)\,'} + n(\alpha+\beta+n+1)\,P_n^{(\alpha,\beta)} = 0$$

se remplaza $\alpha=\beta=-\frac{1}{2}$ y da

$$(1-x^2)\,T_n'' - (0+(2-1)x)\,T_n' + n(-1+n+1)\,T_n = 0$$

$$\boxed{(1-x^2)\,T_n'' - x\,T_n' + n^2\,T_n = 0} \qquad (1.2)$$

2) Para poner (1.2) en forma autoadjunta se multiplica por

$$h(x) = \frac{1}{1-x^2}\exp\left[\int^x \frac{-x'}{1-x'^2}\,dx'\right]$$

$$(1-x^2)\,h(x) = \exp\left(\int^x \frac{-x'}{1-x'^2}\,dx'\right)$$

La ecuación queda:

$$\exp\left[\int^x \frac{-x'}{1-x'^2}\,dx'\right]\frac{d^2}{dx^2}(T_n) - \underbrace{\frac{x}{1-x^2}\exp\left(\int^x \frac{-x'}{1-x'^2}\,dx'\right)}\frac{dT_n}{dx} + n^2\,T_n(x)\,\frac{1}{1-x^2}\exp\int^x \frac{-x'}{1-x'^2}\,dx' = 0$$

$$\underbrace{\exp\left(\int^x \frac{-x'}{1-x'^2}\,dx'\right)\frac{d}{dx}(T_n') + \frac{d}{dx}\left(\exp\int^x \frac{-x'}{1-x'^2}\,dx'\right)T_n'} + \frac{n^2\,T_n}{1-x^2}\exp\int^x \frac{x'}{1-x'^2}\,dx' = 0$$

$$\boxed{\frac{d}{dx}\left(\exp\left(\int^x \frac{-x'}{1-x'^2}\,dx'\right)T_n'\right) + n^2\,T_n(x)\,h(x) = 0}$$

$$\int^x \frac{-x'}{1-x'^2}\,dx' = \int^x -\frac{1}{2}\,\frac{d(x'^2)}{1-x'^2} \underset{u=x'^2-1}{=} \frac{1}{2}\int^{x^2-1}\frac{du}{u} = \frac{1}{2}\ln|x^2-1| = \ln(1-x^2)^{1/2}$$

$u = x'^2 - 1$

$$\exp\left[\int^x \frac{-x'}{1-x'^2}\,dx'\right] = (1-x^2)^{1/2} \quad \text{y} \quad h(x) = \frac{(1-x^2)^{1/2}}{1-x^2} = (1-x^2)^{-1/2}$$

La ecuación diferencial queda:

$$\boxed{\frac{d}{dx}\left((1-x^2)^{1/2}\,T_n'(x)\right) + n^2 (1-x^2)^{-1/2}\,T_n(x) = 0}$$

El valor propio es $\boxed{\lambda_n = n^2}$

Las condiciones de frontera son $\boxed{T_n(1) = 1}$ y $\boxed{T_n(-1) = (-1)^n}$

Esta última se deduce de la paridad $T_n(-x) = (-1)^n T_n(x)$ debida a que $\rho(-x) = \rho(x)$.

3) $\rho(x) = (1-x^2)^{-1/2}$ y $p(x) = (1-x^2)^{1/2}$

$$dt = \sqrt{\frac{\rho(x)}{p(x)}}\,dx = \left(\frac{(1-x^2)^{-1/2}}{(1-x^2)^{1/2}}\right)^{1/2} dx = \left(\frac{1}{1-x^2}\right)^{1/2} dx$$

$$t = \int^x \frac{1}{\sqrt{1-x'^2}}\,dx' \underset{\substack{x = \cos\theta \\ dx' = \sin\theta\, d\theta}}{=} \int^{\arccos x} \frac{\sin\theta}{\sin\theta}\,d\theta = \theta$$

El cambio de variable es $x = \cos t \qquad 1-x^2 = (\sin t)^2$

$$w(t) = \left(\rho(x)\,p(x)\right)^{1/4} u(x) = u(x)$$

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La ecuación se transforma en: $\quad \dfrac{d}{dx} = \dfrac{dt}{dx}\dfrac{d}{dt} = \left(\dfrac{1}{1-x^2}\right)^{1/2}\dfrac{d}{dt} = \dfrac{1}{\sin t}\dfrac{d}{dt}$

$$\frac{d}{dx}\left[\left(1-x^2\right)^{1/2}\frac{d}{dx}u(x)\right] = \frac{1}{\sin t}\frac{d}{dt}\left[\sin t\,\frac{1}{\sin t}\frac{d}{dt}w(t)\right]$$

$$= \frac{1}{\sin t}\frac{d^2}{dt^2}w(t)$$

y $\quad n^2\left(1-x^2\right)^{-1/2}u(x) = n^2\dfrac{1}{\sin t}\,w(t)$

$$\frac{1}{\sin t}\frac{d^2}{dt^2}w(t) + n^2\frac{1}{\sin t}\,w(t) = 0$$

$$\boxed{\frac{d^2}{dt^2}w(t) + n^2\,w(t) = 0}$$

4) $\quad w(t) = A\cos(nt) + B\sin(nt)$

La solución debe ser un polinomio en $x = \cos t$ por lo tanto $B = 0$

y $\quad u(1) = 1 \quad$ da $\quad A = 1$: $\quad w(t) = \cos(nt)$

5) $\quad \boxed{w(t) = \cos(nt) = T_n(x) \qquad \text{con} \quad x = \cos t}$

6) $\quad$ Con $\theta = t \qquad e^{in\theta} = (\cos\theta + i\sin\theta)^n = \displaystyle\sum_{k=0}^{n}\binom{n}{k}(\cos\theta)^{n-k}(i)^k(\sin\theta)^k$

$$e^{in\theta} = \cos n\theta + i\sin(n\theta) = T_n(x) + i\sin(n\theta)$$

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$T_n(x)$ es la parte real de $e^{in\theta}$: son las potencias pares de la suma, para $k = 2l$, l entero

$$T_n(x) = \sum_{l=0}^{E(n/2)} \binom{n}{2l} (\cos\theta)^{n-2l} (-1)^l (\sin\theta)^{2l}$$

con $E(n/2)$ = parte entera de $n/2$

Usando $x = \cos\theta$ y $(\sin\theta)^2 = 1 - x^2$

$$\boxed{T_n(x) = \sum_{l=0}^{E(n/2)} (-1)^l \binom{n}{2l} x^{n-2l} (1-x^2)^l}$$

II. Integrales gaussianas y polinomios de Hermite

1) La transformada de Fourier de una gaussiana es una gaussiana:

$$f(t) = e^{-t^2}$$

$$\hat{f}(\nu) = \int_{-\infty}^{+\infty} e^{-t^2 + 2i\pi\nu t} dt$$

$$= \int_{-\infty}^{+\infty} e^{-(t - i\pi\nu)^2} dt \; e^{(i\pi\nu)^2}$$

$$= \int_{-\infty + i\pi\nu}^{+\infty + i\pi\nu} e^{-u^2} du \; e^{-\pi^2\nu^2}$$

$$\boxed{\hat{f}(\nu) = \sqrt{\pi} \; e^{-\pi^2\nu^2}}$$

2) $$\frac{d^n}{d\nu^n}\hat{f}(\nu) = \sqrt{\pi}\,\frac{d^n}{d\nu^n}\left(e^{-(\pi^2\nu^2)}\right)$$

$$= \sqrt{\pi}\,\frac{\pi^n}{\pi^n}\frac{d^n}{d\nu^n}\left(e^{-(\pi^2\nu^2)}\right)$$

$$\boxed{\frac{d^n}{d\nu^n}\hat{f}(\nu) = \sqrt{\pi}\,\pi^n(-1)^n e^{-\pi^2\nu^2} H_n(\pi\nu)}$$

3) Por las propiedades de la transformada de Fourier sabemos que

$$\frac{d^n}{d\nu^n}\hat{f}(\nu) = \int_{-\infty}^{+\infty}(-2i\pi t)^n f(t)\, e^{-2i\pi\nu t}\,dt$$

$$\frac{d^n}{d\nu^n}\hat{f}(\nu) = \int_{-\infty}^{+\infty}(-2i\pi)^n t^n e^{-t^2-2i\pi\nu t}\,dt$$

Poniendo $x = \pi\nu$ tenemos

$$\frac{d^n}{d\nu^n}\hat{f}(\nu) = \int_{-\infty}^{+\infty}(-2i)^n\pi^n t^n e^{-t^2-2ixt}\,dt = \sqrt{\pi}\,\pi^n(-1)^n e^{-x^2} H_n(x)$$

$$\boxed{\int_{-\infty}^{+\infty} t^n e^{-t^2-2ixt}\,dt = \sqrt{\pi}\, i^n\, 2^{-n} e^{-x^2} H_n(x)}$$

4) Con $x = 0$ $\boxed{M_n = \sqrt{\pi}\,\frac{i^n}{2^n}\,H_n(0)}$

Si n es impar $H_n(0) = 0$ $\Gamma_n = 0$

5) Los polinomios de Hermite satisfacen

$$-\frac{1}{2}\frac{d^2}{dx^2}\left(e^{-x^2/2} H_n(x)\right) + \frac{1}{2}x^2 e^{-x^2/2} H_n(x) = \left(n+\frac{1}{2}\right) e^{-x^2/2} H_n(x)$$

Sea $\psi(x) = e^{-x^2/2} H_n(x)$ y $\hat{\psi}(k) = \int_{-\infty}^{+\infty} e^{-ikx} \psi(x)\, dx$

$$-\frac{1}{2}\frac{d^2}{dx^2}\left(\psi(x)\right) + \frac{1}{2}x^2 \psi(x) = \left(n+\frac{1}{2}\right)\psi(x)$$

La transformada de Fourier de esta ecuación es:

$$-\frac{1}{2}(ik)^2 \hat{\psi}(k) - \frac{1}{2}\frac{d^2\hat{\psi}}{dk^2}(k) = \left(n+\frac{1}{2}\right)\hat{\psi}(k)$$

$$\boxed{-\frac{1}{2}\frac{d^2}{dk^2}\hat{\psi}(k) + \frac{1}{2}k^2\hat{\psi}(k) = \left(n+\frac{1}{2}\right)\hat{\psi}(k)}$$

Es la misma ecuación. Como las soluciones deben ser proporcionales se concluye que $\hat{\psi}(k) = c\,\psi(x)$

$$\boxed{\mathcal{F}\left(e^{-x^2/2} H_n(x)\right)(k) = c\, e^{-k^2/2} H_n(k)}$$

Se deduce c mirando el coeficiente de grado más alto $a_{nn} = 2^n$

$$\mathcal{F}\left(e^{-x^2/2} a_{nn} x^n\right)(k) = a_{nn}\int_{-\infty}^{+\infty} e^{-x^2/2} x^n e^{-ikx}\, dx$$

Con $z = \frac{x}{\sqrt{2}}$ $\longrightarrow$ $= 2^n \int_{-\infty}^{+\infty} e^{-z^2} (\sqrt{2}z)^n e^{-ik\sqrt{2}z}\, dz\sqrt{2}$

$$= 2^n 2^{n/2} \int^{+\infty} e^{-z^2} z^n e^{-2iz\frac{k}{\sqrt{2}}}\, dz\,\sqrt{2}$$

$$= 2^n 2^{n/2} \int_{-\infty}^{+\infty} e^{-z^2} z^n e^{-2iz\frac{k}{\sqrt{2}}} dz \sqrt{2}$$

$$= 2^n 2^{n/2} \sqrt{\pi}\, i^n 2^{-n} e^{-(k/\sqrt{2})^2} H_n(k/\sqrt{2}) \sqrt{2}$$

$$a_{nn} \mathcal{F}\left(e^{-x^2/2} x^n\right)(k) = 2^{n/2} \sqrt{2\pi}\, i^n e^{-k^2/2}\left(a_{nn}\left(\frac{k}{\sqrt{2}}\right)^n + \cdots\right)$$

$$\mathcal{F}\left(e^{-x^2/2} a_{nn} x^n\right)(k) = i^n \sqrt{2\pi}\, i^n \left(a_{nn} k^2 + \cdots\right)$$

Comparando con

$$\mathcal{F}\left(e^{-x^2/2} H_n(x)\right)(k) = c\, e^{-k^2/2} H_n(k)$$

se deduce que $c = i^n \sqrt{2\pi}$

Finalmente queda:

$$\boxed{\int_{-\infty}^{+\infty} e^{-x^2/2} H_n(x)\, e^{-ikx} dx = i^n \sqrt{2\pi}\, e^{-k^2/2} H_n(k)}$$