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Hume's Principle and the Number of all Objects

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As is now well known, the (second-order) Peano postulates may be derived in a consistent second-order system with only one non-logical primitive expression and one non-logical axiom.[1] The non-logical primitive expression, which is intended to mean 'the number of' and which I shall write as 'N', operates on an open sentence with one free variable to produce a closed singular term, so that the result of applying it to the open sentence 'Fx' is '$Nx{:}Fx$', or 'NF' for short. The non-logical axiom says that the number of Fs is identical with the number of Gs precisely when there is a one-to-one correspondence between the Fs and the Gs. In symbols: $\forall F \forall G(NF = NG \leftrightarrow F \approx G)$. George Boolos dubbed this axiom *Hume's principle*, after his claim that "when two numbers are so combin'd, as that the one has always a unite answering to every unite of the other, we pronounce them equal" (*Treatise* I.III.I para.5; cfr. Boolos 1987, p. 6). Boolos also used the name *Frege arithmetic* to stand for the system obtained by adjoining Hume's principle to axiomatic second-order logic, and the name *Frege's theorem* for the result that 'zero', 'immediately precedes' and 'natural number' can be defined in the language of Frege arithmetic in such a way that the second-order Peano postulates may be proved in that system from those definitions. Of course, the historical Frege did not derive his versions of the Peano postulates in the consistent system that Boolos calls Frege arithmetic. Rather, his derivations were cast in the inconsistent system that is obtained by adjoining to axiomatic second-order logic the notorious Basic Law V. But because the courses-of-values regulated by Basic Law V play so narrowly circumscribed a rôle in Frege's proofs of his basic laws of arithmetic, it is straightforward to recast those proofs as derivations in Frege arithmetic.[2] So the name 'Frege's theorem' is in order.

Significant as he believed Frege's theorem to be, Boolos came not to praise Hume's principle but to cast doubt upon it. Crispin Wright had invoked the theorem to vindicate the epistemological kernel of Frege's logicism. According to Wright, Hume's principle—although not a logical truth—is still a statement whose truth we can come to know through an analysis of the concept of a cardinal number. Since, by Frege's theorem, the Peano postulates may be deduced from it, all the theorems of axiomatic second-order Peano arithmetic may be known through logical deduction from a product of conceptual analysis. Accordingly, "intellection of the form of temporal or spatial 'intuition' has no essential part to play in the epistemology of number theory" (Wright 1983, p. 154), so that Frege was right, and Kant was wrong, about the epistemology of arithmetic. Boolos had much to say against this account of the significance of Frege's theorem; but for present purposes only one of his criticisms interests me. If we can know Hume's principle—whether through conceptual analysis or otherwise—then it really ought to be true. And yet:

> not only do we have no reason for regarding Hume's principle as a truth of logic, it is doubtful whether it is a truth at all. As the existence of a number, 0, belonging to the concept *not-self-identical* is a consequence of Hume's principle, it also follows that there is a number belonging to the concept *self-identical* [i.e. $Nx{:}x=x$], a number that is the number of things [*sc*. objects] that there are. Hume's principle is no less dubious than any of its consequences, one of which is the claim, uncertain at best, that there is such a number.[3]

In this paper, I wish to consider whether this consequence really is dubious and, if it is, how Hume's principle might best be emended so that it no longer entails it.

I

Boolos gives the following reason for doubting whether there is such a thing as the number of all objects, or, as he follows Wright in calling it, *anti-zero*:

> On the definition of $\leq$, according to which $m \leq n$ iff $\exists F \exists G(m=NF \wedge n=NG \wedge$ there is a one-one map of F into $G)$, anti-zero would be a number greater than any other number. Now the worry is this: *is* there such a number as anti-zero? According to Zermelo-Fraenkel set theory, there is no (cardinal) number that is the number of all the sets there are. The worry is that the theory of number we have been considering, Frege arithmetic, is incompatible with Zermelo-Fraenkel set theory plus standard definitions, on the usual and natural readings of the non-logical expressions of both theories (Boolos 1997, p. 260, with incidental changes in notation).

What, though, are the "standard definitions" that render Zermelo-Fraenkel set theory incompatible with the existence of anti-zero? Crucial among them is the ZF set-theorist's conception of cardinals as a species of ordinal. He defines the cardinal number of Fs to be the least ordinal number whose members are equinumerous with the Fs, i.e. to be the least ordinal α for which there exists a one-to-one correlation between the *F*s and the members of α. (The members of an ordinal are precisely the ordinals less than it.) It follows from this definition, in tandem with the axiom of replacement, that there is a cardinal number of *F*s only when there is a set of *F*s.[4] That axiom says that for any set α and any function *f*, the image under *f* of the members of α will constitute a set; and when there is an ordinal whose members correlate one-to-one with the *F*s, the *F*s will be the image under the correlation of the ordinal's members, which certainly constitute a set. Boolos is quite right to say, then, that ZF disallows a cardinal number of sets. If there were such a thing, there would have to be a set of all sets, and hence by *Aussonderung* a set of non-self-membered sets; yet the reasoning of Russell's paradox constitutes a proof in ZF that there is not. For a similar reason, there can be no cardinal number of ordinals. If there were, then there would have to be a set of all ordinals, and the reasoning of Burali-Forti's paradox constitutes a proof in ZF that there is not. Finally, Boolos is also right to suppose that these theorems of ZF disallow anti-zero. If there were a number of all objects, there would have to be a set of all objects and hence, by *Aussonderung*, a set of all sets and a set of all ordinals, since each set and each ordinal is an object.

Powerful as this reasoning may appear, however, an adherent of Frege arithmetic is in any case committed to rejecting its starting point. Frege arithmetic, we may presume, is intended to make explicit Frege's conception of number, but that conception cannot tolerate the identification of cardinals with ordinals. Frege held that a certain range of applications is partially constitutive of each different kind of number. Cardinal numbers (including the natural numbers) are applied in answering 'How many?' questions, whereas "numbers that give a measure" (including the reals) are applied in answering 'How much?' This, indeed, is Frege's ground for his well known denial (in the second volume of his *Grundgesetze der Arithmetik*) that a natural number can ever be identified with a real number:

> It is not possible to enlarge the realm of cardinal numbers (*Anzahlen*) to that of the real numbers; they are wholly distinct domains. The cardinal numbers answer the question 'How many objects of a given kind are there?', whereas the real numbers can be regarded as numbers giving a measure, saying how large a quantity is as compared with a unit quantity.[5]

Less widely appreciated, however, is his parallel argument against identifying cardinals with ordinals. Discussing Cantor's notion of number in *Die Grundlagen der Arithmetik*, he remarks how

> in ordinary use the words 'cardinal number' and the question 'How many?' contain no reference to any fixed order. Cantor's numbers, by contrast, answer the question: 'the how-manyeth member of the succession is the last member?' (Frege 1884 §85, p. 98).

Or, more generally and more idiomatically, they answer the question ‘Where does this object stand in that ordering?’ Questions of this latter kind differ from ‘How many?’ questions just as basically as do questions of quantity, yet it is they which ordinal numbers are invoked to answer. So, if Frege’s argument provides a good reason for saying that the cardinals constitute a distinct domain from the “numbers that give a measure”, then there is also good reason to deny the identification of cardinals with ordinals on which Boolos’s attack on anti-zero rests.

I shall return to the relationship between cardinals and ordinals in the next section. But even if Frege is right about their distinctness, so that we must reject the ZF definitions which entail the non-existence of anti-zero, there are at least two reasons, of a rather more philosophical kind, for agreeing with Boolos that the existence of anti-zero is at best uncertain. The first of these is suggested by Benacerraf’s argument in “What Numbers Could Not Be” (1965). There are certainly two natural numbers less than 2; and there are certainly two von Neumann ordinals less than $\{\emptyset, \{\emptyset\}\}$. But if there is to be a uniquely correct answer to the question ‘How many objects are either natural numbers less than 2 or von Neumann ordinals less than $\{\emptyset, \{\emptyset\}\}$?’ there will have to be correct answers to the four questions ‘Is the number 0 identical with the von Neumann ordinal $\emptyset$?’,…, ‘Is the number 1 identical with the von Neumann ordinal $\{\emptyset\}$?’ More generally, the existence of a uniquely correct answer to the question ‘How many objects are there?’ will require the existence of a uniquely correct answer to any question of identity that may be raised about objects that are initially identified as belonging to different sorts. It will require, in other words, not merely that there should be a correct answer to the question ‘How many *A*s that are *F* are there?’, for any count noun ‘*A*’ and any predicate ‘*F*’ that imposes a determinate restriction on *A*s; but also that there should be a uniquely correct answer to any question in the form ‘Is this *A* identical with or distinct from that *B*?’ For any such question must have an answer if there is to be an answer to the question whether the *B* in question has already been counted among the *A*s. It is, though, highly doubtful whether there is always a correct answer to a question of identity that cuts across sorts in this way. Benacerraf gives reasons for doubting whether there is a correct answer to any of our four questions relating numbers and sets, and cognate examples can be constructed for kinds of concrete object. There is an exact number of books in my college room. There is also an exact number of mereological fusions of book-boards, book-spines, and book-pages in my room. But if there is to be a number of things in my room which are either books or fusions of book-parts, then there will have to be answers to such questions as whether my copy of Frege’s *Grundlagen* is strictly identical with the fusion of its boards, spine and pages. But the claim that there are such answers is highly doubtful. Accepting it involves accepting the view (powerfully attacked by Carnap) that there will always be a “fact of the matter” whether a putative ontological reduction succeeds.

Curiously, Wright shows himself sympathetic to such doubts. He discusses a case in which the brain of a man, Jones, is somehow divided and transplanted so as to animate a pair of decorticate human bodies, *Brown* and *Black*, and in which Jones's body is then destroyed. It is then clear, he says, that two biological organisms survive. But in order to answer the question 'How many objects are there, identical with either Jones, *Brown* or *Black*?' we should have to

> say which among various organisms still surviving, to wit *Brown* and *Black*, Mr Jones now is—if, indeed, he is with us at all. It is evident that the problem is intractable: there is not enough information in the description of the case to decide the matter, and it is intuitively clear that one could provide as rich a description of the case as one liked without building in any basis for decision (1983, p. 125).

The moral Wright draws is that the notion of a sortal or countable concept needs to be relativised. A concept is not sortal or non-sortal *tout seul*. Rather, concepts

> are sortal only in relation to each other. A set of concepts, we can say, are mutually sortal only if it is a determinate matter which, if any, share which of their instances. Such a set would be, for example, {tree, person, natural number, molecule, direction}. Now: what good reason is there to think that all the concepts which we can combine into mutually sortal groups can be combined into a single such comprehensive group? (1983, p. 124).

On Wright's view there is no such reason. The set {man, natural number} is sortal; and so is the set {biological organism, von Neumann ordinal}; but the union set {man, biological organism, natural number, von Neumann ordinal} is not. The example shows how an unrestricted amalgamation of sortal sets of concepts goes beyond anything needed to sustain our ordinary practices of counting. It is, however, needed to sustain the existence of anti-zero.[6] This casts doubt on Frege arithmetic as a formalisation of those ordinary practices. For the formalisation incurs a dubious commitment which the practice it systematises escapes.

Some philosophers will be unimpressed by this first philosophical argument for doubting the existence of anti-zero. It turns on finding cases of indeterminacy of identity, and while examples such as Benacerraf's may well be cases in which there is no possible *basis* for a decision about identity, the philosophers I have in mind will deny that these undecidable questions of identity lack answers. For these philosophers advance general logical grounds for deeming indeterminacy of identity to be incoherent. Wiggins (1986), for example, refining the argument of Evans 1978, has attempted to prove that if things are identical, they must be determinately identical; and Williamson (1996) has tried to derive the corresponding result for distinctness. On their view, then, the absence of any conceivable basis for deciding certain questions of identity simply reflects a deep epistemic limitation. It does not provide a ground for saying that those questions actually lack answers and accordingly does not provide a ground for denying that there is any answer to the question 'How many objects are there?'

The principles on which these derivations rely are of course controversial, but a second reason for doubting the existence of anti-zero speaks even to those who accept them. Even if there were a determinate relation of identity/distinctness across instances of different kinds, and even if a number were attached to each particular kind of object (i.e. even if there were an exact number of sets, and of numbers, and of zebras, etc.), it still would not follow that there is an exact number of objects. For there could fail to be such a thing by virtue of there being no exact number of *kinds* of object. In the relevant sense, a kind of object exists precisely when there is a principle for counting its instances, and the claim that there is a determinate collection of such principles is highly doubtful. There is surely no determinate collection of counting principles that people like us could learn to go by, for it is indeterminate how like us the relevant people must be, and also indeterminate how much conceptual or linguistic change this 'could' permits. Accordingly, the claim that there *is*, nevertheless, a determinate collection of counting principles whose boundaries are settled quite independently of our ability to use them is a strong realist claim unsupported by anything in our actual practice.

Of course, nothing in the previous four paragraphs constitutes an argument for the actual non-existence of anti-zero, and hence for the falsity of Hume's principle. Like Boolos, I aim only to show that it is "uncertain at best". But the uncertainty is quite sufficient to motivate the project that I wish to undertake. Namely: to see whether there is not some weaker thesis—in the same family as Hume's principle—which entails the Peano postulates without entailing the existence of anti-zero. Were there such a thesis, and were it knowable simply through conceptual analysis of the notion of cardinal number, a Fregean epistemology for cardinal arithmetic would not be hostage to the existence of anti-zero.

II

Some philosophers will readily agree that the existence of anti-zero is problematical but will deny that Hume's principle is the source of the problem. The principle, they will observe, is committed to the existence of a number of all objects only when its first-order variables are permitted to range over all objects. Frege himself did permit this, of course, but the philosophers I have in mind withhold such permission on grounds that are quite independent of the existence of anti-zero. According to Michael Dummett, for example, "the one lesson of the set-theoretical paradoxes which seems quite certain is that we cannot interpret individual variables in Frege's way, as ranging simultaneously over the totality of all objects which could meaningfully be referred to or quantified over" (1981, p. 567). First-order variables, on this view, must be restricted to ranging over some antecedently circumscribed domain; and once they are so restricted Hume's principle will not entail the existence of anti-zero.

There are certainly some ways of restricting the range of the first-order variables in Hume's principle which render it true under its intended interpretation. If, for example, those variables are confined to ranging over the domain $\{0, 1, \ldots, n, \ldots, \aleph_0\}$, then the intended reference of '$Nx{:}x=x$'—viz. $\aleph_0$—falls within the domain, and it is easy to verify that Hume's principle then comes out true under its intended interpretation. (Indeed, the Burgess/Boolos proof that Frege arithmetic is consistent relative to analysis starts from this observation.[7]) However, the imputation of paradox to systems whose objectual variables range unrestrictedly lacks convincing support. Richard Cartwright (1994) has argued powerfully that the set-theoretical paradoxes arise, not from the use of unrestricted variables *per se*, but from their use in tandem with what he calls the "All-in-One Principle"—the thesis, namely, that variables may range over certain objects only when there is some one thing to which those objects belong. The proper "lesson" of the paradoxes is the falsity of this principle; once it is rejected, unrestricted quantification is permissible. What is more, such quantification ought to be permitted on pain of rendering certain patent truths inexpressible: "Absolutely every object is self-identical", "Absolutely every object is mortal if human", etc. David Lewis, indeed, notes that paradox threatens the claim that the quantifiers *cannot* be understood unrestrictedly: a "mystical censor [who] stops us from quantifying over absolutely everything without restriction…violates his own stricture in the very act of proclaiming it" (1991, p. 68). In the light of these observations, it seems misguided to allow Frege arithmetic to escape its problematical commitment to anti-zero by imposing a blanket ban on unrestricted quantification. For the problem that anti-zero presents does not seem to arise from the unrestricted range of the theory's *objectual* variables, but rather from the assumption—which is built into the use that Hume's principle enjoins for the term-forming operator 'N'—that a cardinal number belongs to every *concept*.

If, however, we are to avoid the problematical commitment by emending Hume's principle in some way, can we be more specific about the feature of the principle that is responsible for that commitment? How can the principle be dissected into logically distinguishable parts? An important first incision has been made by Dummett. He abhors the term 'Hume's principle' because it

> blurs a vital distinction between two quite distinct principles:
> (1) the definition of 'just as many…as' (of 'equinumerous'):
> There are just as many *F*s as *G*s iff there is a one-one map of the *F*s on to the *G*s
>
> and
>
> (2) the equivalence (EN)—the abstraction principle proper:
> The number of *F*s = the number of *G*s iff there are just as many *F*s as *G*s (Dummett 1998, p. 386).

Certainly, if we are to continue to use the term, we must bear in mind that Hume's principle is the conjunction of these quite different theses. It is one thing to advance a definition (or a conceptual analysis) of the notion of there being exactly as many as. It is another to move from the premiss that there are exactly as many *F*s as *G*s to the conclusion that there is an object, the number of *F*s, with which the number of *G*s is identical.[8]

Neil Tennant, the writer who has had most to say about the problem we are addressing, diagnoses it as stemming from Dummett's principle (2). The one non-logical primitive in the language of Frege arithmetic, he reminds us, is the singular term-forming operator '*N*'; and because the language contains semantically complex terms, we cannot assume

> that all singular terms denote just by virtue of being grammatically well-formed. Instead, we have to take seriously the possibility of 'empty', or non-denoting singular terms, even when they are grammatically well-formed. What we need, in short, is a *free* logic (Tennant 1997, p. 311)

—i.e., a logic in which claims in the form ⌜*t* exists⌝ may be false for some singular terms *t*.[9] And while the left-to-right half of principle (2)—which Tennant labels (N1)—is correct as it stands, he argues that the conditional from right to left is not:

> (N2) The number of *F*s is identical to the number of *G*s if there are exactly as many *F*s as *G*s
>
> ...is not analytically true as it stands. *F* and *G* might have such vast extensions that, although they might be in one-to-one correspondence, they nevertheless enjoy no number as their cardinality. The analytically true claim is rather
>
> (N2*) The number of *F*s is identical to the number of *G*s if (there are exactly as many *F*s as *G*s and the number of *F*s exists and/or the number of *G*s exists) (Tennant 1997, p. 313).

Even when supplemented by Frege's definition of 'exactly as many as', the conjunction of (N1) and (N2*)—a conjunction I shall label (2*)—does not yield the Peano postulates. Further principles asserting or entailing the existence of numbers are needed. Tennant shows, however, that the postulates may be derived when principles (1) and (2*) are combined with the following two existence principles:

(A) if there are no *F*s, then the number of *F*s exists;
(B) if the number of *F*s exists and there is exactly one more *G* than there are *F*s, then the number of *G*s also exists.[10]

('There is exactly one more *G* than there are *F*s' is defined to mean: 'There is an object *x* which is a *G* and for which the *F*s are equinumerous with the *G*s distinct from *x*'.) Principle (A) entails the existence of zero; and principle

(B), which Tennant dubs the *ratchet principle*, may then be applied repeatedly to establish the existence of each positive natural number. Those principles, however, do not entail the existence of anti-zero. When combined with the suggested existence principles, then, principles (1) and (2*) appear to offer the benefits of Hume's principle without the uncertainties.

If those benefits are to include a vindication of Frege's epistemology for elementary arithmetic, however, then the existence principles needed to derive the Peano postulates must themselves be knowable without any appeal to "intuition". But are they? Tennant himself supposes that they are. He describes his existence principles as "core *conceptual* truths about natural numbers" (1997 p. 319, with emphasis added), and he attempts to justify the claim that the ratchet principle is analytic by explaining that it

> is really toothless ontologically. All it expresses is the thought that if one has gone so far as to acknowledge the existence of any one natural number, then there is no reason to refuse to recognise the 'next' number. That seems reasonable: not even the nominalist opponent wishes to visit on the Platonist a prematurely truncated initial segment of the natural number series, denying the Platonist all (and only) the numbers after some allegedly 'final' one (1997, p. 319).

I do not see, however, that this establishes that the ratchet principle is analytic in the epistemologically relevant sense of being knowable by logical deduction from the deliverances of conceptual analysis. It may well be hard to find anybody who denies it; hard even to find somebody who denies that it is "reasonable" to accept it. The question, however, is whether there are reasons for accepting it that make no appeal (however surreptitiously) to any deliverances of "intuition".

It seems, moreover, that there is only one way to establish an affirmative answer to this question on a really secure basis. Namely: by formulating a numerical existence principle that may be shown to follow from a conceptual analysis of the notion of cardinal number; and then by giving a strictly logical deduction of the ratchet principle from this principle. And it is clear on reflection that an existence principle logically grounded in a conceptual analysis will be needed in *any* vindication of Frege's epistemology for elementary arithmetic in the context of a free logic. His major claim in this area is that elementary arithmetical theorems may be proved "from general logical laws and from definitions" (Frege 1884 §3, p. 4). Vindicating this claim will involve giving a strictly logical derivation of the Peano postulates from axioms that are "definitions" in the sense of being knowable through an analysis of the notion of number and through that alone. When the derivation is to take place in a free logic, these general laws will have to include a principle of numerical existence. So, vindicating Frege's epistemology requires finding an existence principle which (i) may be known through conceptual analysis; and which (ii) combines with principles that may be known on similar grounds—such as (1) and (2*)[11]—to entail the Peano postulates. Tennant's work shows that condition (ii) will be met if we can find an existence principle which entails both principle (B) and the existence of zero. But there may of course be other ways of meeting it.

What might a suitably general existence principle be? What is it for there to be such a thing as the cardinal number of *F*s? The connection that Frege posits between cardinals and 'How many?' questions suggests the following first shot at an answer: there is such a thing as the number of *F*s if and only if the question 'How many *F*s are there?' has a uniquely correct exact answer. That is what it is for there to be a unique number of *F*s; for cardinal numbers are nothing other than things invoked to keep the tally as we give exact answers to various 'How many?' questions.[12]

This account of numerical existence may appear to undo the use we are making of a free logic by ensuring that the number of *F*s exists for any *F*. For let us assume the truth of Dummett's principle (1)—namely, that there are exactly as many *F*s as *G*s iff there is a one-one map of the *F*s on to the *G*s. Formally, a relation *R* maps the *F*s one-one on to the *G*s iff

$$\forall x[Fx \rightarrow \exists y(Gy \wedge Rxy)] \wedge \forall y[Gy \rightarrow \exists x(Fx \wedge Rxy)] \wedge \\ \forall x \forall u \forall y \forall v\, [Rxy \wedge Ruv \rightarrow (x{=}u \leftrightarrow y{=}v)].$$

Now it is a theorem of first-order logic that identity maps the *F*s one-one on to themselves, so there will always exist a one-one map of the *F*s on to the *F*s. If principle (1) is correct, then, there will always be exactly as many *F*s as there are *F*s. And this might seem to entail that there will always be a correct exact answer to the question 'How many *F*s are there?' Namely: precisely as many as there are *F*s.

Does such a response really qualify, though, as an exact answer to the question? A passage in *Grundlagen* itself suggests a strong reason for supposing that it does not. "The numbers", Frege writes in §10, "are related to one another quite differently from the way in which the individual specimens of, as it might be, a species of animal are, for it is in their nature to be arranged in a definite order of precedence" (Frege 1884, p. 15). The context leaves it somewhat unclear whether Frege meant to propound this doctrine for cardinal numbers in general or simply for finite numbers. Whatever his intentions were, however, it is plausible to demand of any satisfactorily exact answer to a 'How many?' question that it *should* indicate exactly how it relates to other possible answers. This is a task which one who responds to the question 'How many *F*s are there?' by saying 'Exactly as many as there are *F*s' signally fails to accomplish. This observation in turn enables us to identify more accurately the feature of Frege's theory of number that is responsible for its commitment to the existence of anti-zero. The problem is not merely the use of a logic that presupposes a reference for each well-formed singular term. For, as we have seen, even when that presupposition is cancelled by deploying a free logic, Hume's principle entails that for any concept *F*, the number of *F*s is identical with the number of *F*s and hence—even on standard free-logical assumptions—that the number of *F*s exists. The deeper problem is that the criterion of numerical identity—and hence of numerical existence—encapsulated in Hume's principle fails to respect the point that it is in the nature of cardinal numbers (and not just of finite cardinals) to be arranged in a definite order.

What, though, are the essential features of that order? It is natural to try to answer this question by formulating conditions on the relation *R* that relates a member of a sequence of number-words (i.e. a sequence of objects suited to provide exact answers to 'How many?' questions) to its immediate successor. In the first place, such a sequence must include a member with no predecessor, for a count has to begin somewhere. That is to say, we must have: (i) $\exists x\ \neg\exists y\ Ryx$. It is also clear that the *R*-relation should be *eindeutig*, or *functional*, for there cannot be more than one right way to continue a count. In symbols: (ii) $\forall x \forall y \forall z\ [Rxy \wedge Rxz \rightarrow y{=}z]$.[13] To be sure, one can imagine an Anglo-French bilingual counting three objects by saying 'one', 'deux', 'three', or 'one', 'two', 'trois', as he pleases. But we can only accept this as a sensible way to count if we understand the tokens of 'deux' or 'two' (for example) as equally admissible instantiations of a hybrid numerical word-type 'deux'/'two'. On this conception of the matter, the *R*-relation that binds the bilingual's numbers-words *will* still be functional; what is unusual about the case is only that a single word-type may be pronounced in two very different ways. Unusual as this may be, it seems to me to be no more untoward than the fact that a given number-word type may be instantiated in different accents, or at different volumes.

A third condition on *R* is indicated by a passage in *Grundgesetze*. Theorem 145 of that work states that no finite cardinal number follows after itself in the number-series (Frege 1893 §113, p. 144). That is to say: no finite cardinal is related to itself by the *strong ancestral* of the relation that relates each cardinal to its immediate successor.[14] And in advertising the significance of this result Frege writes as follows:

> The importance of this proposition becomes clearer through the following considerations. If we determine the number belonging to a concept $\Phi(\xi)$—or, as one ordinarily says, if we count the objects falling under the concept $\Phi(\xi)$—then we successively co-ordinate these objects with the number-words from 'one' up to a number-word '*N*'. This number-word is determined through the co-ordinating relation's mapping the concept $\Phi(\xi)$ into the concept "member of the series of number-words from 'one' to '*N*' '' and the converse relation's mapping the latter concept into the former. '*N*' then designates the number sought; i.e. *N* is this number. This process of counting may be carried out in various ways, since the co-ordinating relation is not fully determined. The question then arises, whether one could reach another number-word '*M*' through another choice of this relation. In that case, it would follow from our results that *M* would be the same number as *N*, and yet at the same time that one of these two number-words would follow after the other, e.g. '*N*' after '*M*'. In that case, [the number] *N* would also follow [the number] *M*

> in the number-series, i.e. *N* would follow itself. Our theorem excludes that possibility for finite cardinals.[15]

It will help to spell out Frege's reasoning in this passage more fully. He begins by observing that the condition for the number word '*N*' to say correctly how many *F*s there are is that there should be a one-one correlation between the *F*s and the number-words that appear no later than '*N*' in the relevant sequence of number-words. (This condition presupposes that number-words are ordered.) As he observes, however, a plethora of relations may establish such a correlation. The number-word 'nine' (in the sequence 'one', 'two', ...) correctly answers the question 'How many symphonies did Beethoven compose?' because the relation π that maps Beethoven's n^{th} symphony (in the order of their composition) to the number-word '*n*' effects a one-one correlation between his symphonies and the number-words from 'one' to 'nine' inclusive. But there is a distinct correlation ρ that maps Beethoven's n^{th} most frequently performed symphony to the number-word '*n*', and "the question arises whether we can reach another number-word ['eight', as it might be] through" this new correlation. If we could, then the sequence of number-words would provide two distinct answers to the question 'How many symphonies did Beethoven compose?' The answer 'nine' would be correct by virtue of correlation π; yet the answer 'eight' would also be correct by virtue of correlation ρ. In such a case, then, there would be no determinate answer—i.e. no unique exact answer—to the question 'How many symphonies did Beethoven compose?'

Now the passage identifies a feature of the relation *R* between a number-word and its immediate successor that excludes this possibility and thereby ensures that number-words ordered by such an *R* provide suitably determinate answers to 'How many?' questions. For what Frege's reasoning brings out is that the expressions related by *R* will not provide suitable answers to 'How many?' questions unless *R*'s strong ancestral R^* is irreflexive. We must, in other words, have: (iii) $\neg\exists x\ R^*xx$. For let us suppose that a speaker replies to the question 'How many *F*s are there?' by saying '*N*'. As Frege notes, in giving this answer, he is affirming the existence of a one-one map from the *F*s on to the number-words from 'one' up to and including '*N*'. Now if the number-word '*N*' were related by R^* to itself, then '*N*' would stand (let us say) in both the *n*th and the *m*th places of the *R*-series. In that case, there would be no telling whether our speaker was deeming the *F*s to be equinumerous with the number-words occupying the first to the *n*th places of that series or with the number-words occupying the first to the *m*th places. Accordingly, the reply '*N*' could not, after all, constitute an unambiguous answer to a 'How many?' question. In general, then, a series of expressions whose ancestral is not irreflexive will fail to supply suitably unambiguous answers to such questions.

As Frege notes, Theorem 145 of *Grundgesetze* excludes that possibility for finite cardinals, but the reasons just given for requiring the relation between a number-word and its successor to meet the three conditions listed are wholly general. If *any* sequence of number-words, whether finite or infinite, is to supply determinate answers to 'How many?' questions, the relation that binds successive members must have a beginning, must be functional, and must have an irreflexive strong ancestral. It is, moreover, highly plausible that these conditions suffice for the members of a sequence to be usable in giving determinate answers to 'How many?' questions. In the first place, it is obviously inessential that the objects that are so used should be things that we ordinarily take to be number-words, or even that they should be things that we ordinarily take to be expressions at all. Demonstrating the *N*th object in a sequence of ordinary material objects that meets conditions (i) to (iii) may constitute a perfectly good way of answering '*N*' to a 'How many?' question, albeit one whose actual employment will often be impractical. The three conditions are, moreover, so strong logically that it seems implausible to require any more of expressions that may be used to record the results of counts. For if some expressions are related by a relation *R* that meets the conditions, then they will be well ordered by *R*'s strong ancestral.[16]

What condition for numerical existence, however, might be derived from these considerations about relations whose relata may be used for counting? A very simple answer runs as follows. Let us say that a concept *F* is a *simple tally* when the objects falling under it are the logical posterity of some object under a relation that meets the three Fregean conditions (i) to (iii). Symbolically:

$$\text{S-tally}(F) \leftrightarrow \exists R(\exists x\, \neg\exists y\, Ryx \wedge \forall x \forall y \forall z\, [Rxy \wedge Rxz \rightarrow y{=}z] \wedge \\ \neg\exists x\, R^{*}xx \wedge \exists x \forall y[Fy \leftrightarrow R^{*}{=}xy]).$$

Now to establish an exact answer to the question 'How many *F*s are there?' is, as Frege says, to establish a one-one correlation between the *F*s and some objects that may be used in counting. And if we suppose that objects usable in counting will be the logical posterity of some object under a relation that meets Frege's conditions, the passage yields the following suggestion. There will be a determinate answer to the question 'How many *F*s are there?' if and only if the concept *F* is equinumerous with an initial segment of a simple tally.

In fact, this suggestion will not do. One problem with it may be dealt with quite simply. In the quoted passage, Frege writes as though 'determining the number belonging to a concept *F*' were synonymous with 'counting the *F*s'. In other writings, however, he gives a compelling reason for supposing that it is not. One can count the *F*s—in the sense of establishing a one-one correlation between the *F*s and the number-words between 'one' and '*N*' inclusive—only if there is at least one *F* to be counted. And yet "in answering 'zero' to the question 'What is the number of Romulus' predecessors on the throne of Rome?'…we are not denying that there is such a number, we are naming it" (Frege 1894, p. 328). Paradoxes result if one construes the answer 'nobody' to the question 'Who preceded Romulus on the throne of Rome?' as the name of a king of Rome. In giving *that* answer, one is denying that there is any such predecessor to be named. The expression ‘zero’, by contrast, “may be used just like all other number words without special precautions” (*ibid.*). So, even though zero is not among what Frege calls the “counting numbers”, the word ‘zero’ does express a possible determination of number. That is to say, it is a possible answer to a ‘How many?’ question rather than a repudiation of that question. The suggested condition for such a question to have an exact answer must, then, be emended at least this much: it will possess one if and only if *F* is either empty or equinumerous with an initial segment of a simple tally.

Another problem with the suggestion cuts much deeper. An object’s logical posterity under a functional relation will not have a cardinality greater than $\aleph_0$, so that this is the greatest cardinality of a simple tally. No doubt this helps to explain why mathematicians call concepts of cardinality no greater than $\aleph_0$ “countable”. Since Cantor, however, they have become accustomed to the idea that there are many greater answers to ‘How many?’ questions which are yet perfectly exact. The philosophical theory of number we are considering contradicts that idea; and few mathematicians will take seriously an account of number that bars so firmly the gates to Cantor’s paradise.

Serious as this difficulty is, I think it may be overcome by a change of focus. In the passage quoted from *Grundgesetze* I §108, the analysis focuses upon the conditions that must be met by the relation between one number-word and its *immediate* successor, if a whole sequence of number-words is to provide exact answers to ‘How many?’ questions. A simple tally was then defined as a concept whose extension comprises objects related to some initial element by such a relation’s weak ancestral. But our earlier reflections upon the activity of counting also yield constraints on the relation of one answer’s *being later than* another (in the relevant ordering of answers) that run parallel to those that constrain the relation of immediately succeeding. And by spelling out the former constraints we can define a more general notion of a tally without invoking ancestrals. In particular, those reflections support the hypothesis that if an ordering of number-words is suited to supply exact answers to ‘How many?’ questions, then the relation *S* of being later than (in the ordering) must be a strict well-ordering. The requirement that *S* should be a linear ordering (i.e. should be transitive and connected) simply generalises the requirement (in the countable case) that a given set of number-words should constitute a sequence. Furthermore, the argument distilled from the passage quoted from *Grundgesetze* shows that *S* must be irreflexive: if a single number-word ‘*N*’ occurred at both the *m*th and the *n*th places, then we would be uncertain how far a count had proceeded before it issued in the answer ‘*N*’. Finally, it is hard to see how a count could proceed in an orderly way unless every non-empty class of number-words had a least member under *S*. For a thinker starts a count by associating an object with the *S*-least number-word (‘one’, as it might be); and he continues by associating a distinct object with the *S*-least number-word that is distinct from 'one' (for example, 'two'). If the objects are suited to providing exact answers to 'How many?' questions, there must be a uniquely correct association to make at each stage; and this requires that each non-empty class of number-words shall have a least member under the relation S.

This suggests the following way of elaborating the idea that there is a non-zero number of Fs just when the Fs can be counted—a way which allows for cardinal numbers greater than $\aleph_0$. Let us say that a concept G is a *generalised tally* iff the objects that fall under it may be strictly well-ordered. Symbolically:

$$\text{Generalised tally}(G) \leftrightarrow \exists R\ (R \text{ is a strict well-ordering on } G)$$

where R is a strict well-ordering on G iff

$$\forall x \forall y \forall z\ [Gx \wedge Gy \wedge Gz \rightarrow ((Rxy \wedge Ryz \rightarrow Rxz) \wedge (Rxy \vee x{=}y \vee Ryx) \wedge \neg\, Rxx \wedge \forall H\ [(\forall w (Hw \rightarrow Gw) \wedge \exists w Hw) \rightarrow \exists u \forall v (Hv \rightarrow Ruv \vee u{=}v)]\ )]$$

The analysis above suggests that the objects falling under a concept G will constitute a range of possible answers to a 'How many?' question just in case G is a generalised tally. And this gives us the needed condition for the question 'How many Fs are there?' to possess an exact non-zero answer. For we shall give such an answer by indicating a particular strict well-ordering R on the Gs, by singling out somehow an object x that falls under G, and by saying that the Fs are equinumerous with the Gs up to and including x under the relation R. This gives the following necessary and sufficient condition for there to be such a thing as the number of Fs:

(C) There is such a thing as the number of Fs iff either F is empty or $\exists G\ \exists R\ \exists x\ (R \text{ is a strict well-ordering on } G \wedge F \approx G{\uparrow}_R\, x)$

where $\forall y\ [(G{\uparrow}_R\, x)y \leftrightarrow (Gy \wedge (Ryx \vee y{=}x))]$. We may express this in words by saying:

(C) There is such a thing as the number of Fs iff F is either empty or equinumerous with a bounded initial segment of some generalised tally.

The requirement that a concept F to which a number belongs should be equinumerous with a *bounded* segment of a tally reflects the fact that in giving an exact non-zero answer x to a 'How many?' question a respondent indicates that all the Fs may be put in correspondence with the members of the tally up and including x. Even if the count could not be completed in a finite time, the count must in this sense be exhaustive.

Proposal (C) rests squarely on the Fregean idea that the existence conditions of cardinal numbers relate directly to their application in answering 'How many?' questions. But in uncovering an order-related element in the notion of an answer to such a question, it gives us the desired room to deny that a number attaches to each and every concept. Certainly, the mere fact that the *F*s may always be correlated one-one with themselves no longer suffices to establish the claim that there will invariably be a number of *F*s. For the fact in question does not entail that the *F*s are equinumerous with a bounded initial segment of some generalised tally.

To discern an order-related element in the notion of an answer to a 'How many?' question is not to identify cardinal numbers with ordinals, as Cantor did and as ZF set-theorists still do. On the contrary; on a view such as Frege's—whereby kinds of number are constituted by their ranges of application—the basic differences between 'How many?' and 'How manyeth?' questions preclude any identification of the numbers that are invoked in answering them.[17] It is, rather, to challenge the idea—which is encapsulated in Hume's principle—that the conditions for the identity of numbers (and hence of their existence) may be explained purely in terms of one-one correspondence. Since the *F*s are always in one-one correspondence with themselves, it is this idea which (as we saw) ultimately underpins the ascription of a number to each *bona fide* concept. In challenging this idea, I am not denying the consistency of the resulting theory of number (i.e. Frege arithmetic, in a free logic), nor its attractiveness as a theoretical simplification of our ordinary arithmetical practice. The fact that it entails the existence of anti-zero, however, shows that the theory's simplicity is bought at the cost of commitments which participation in that practice does not incur. Moreover, the alternative arithmetical theory comprising principles (1), (2*) and (C)—however unfamiliar it may be—arises just as naturally as Frege arithmetic when we try to regiment our ordinary arithmetical practice. Any such regimentation will be a theoretical projection from the judgements that ordinary thinkers make about the finite cardinals, and from the rules of inference that they apply to those judgements. Once the concept of being well ordered is explained to them, such thinkers will take it for granted that the finite cardinals compose a well ordered series. In taking equinumerosity with an initial segment of a well ordered series to be the general criterion for numerical existence, then, principle (C) embodies a theory of cardinal number that is just as natural a projection from our ordinary arithmetical practice as is Hume's principle.

Our discussion also casts some light on Frege's relations with another of his eminent contemporaries. I have been taking more or less for granted the truth of Dummett's principle (1)—that there are exactly as many *F*s as *G*s iff there is a one-one map from the *F*s on to the *G*s. It ought to be observed, however, that there is now a ground for doubting this. As we have seen, there is always a one-one map from the *F*s on to the *F*s, so if principle (1) were correct it ought always to be true to say 'There are exactly as many *F*s as there are *F*s'. One might, however, reasonably doubt whether this *is* always true. For it is far from silly, or even eccentric, to insist that no statement in the form 'There are exactly as many *F*s as *G*s' can be true unless there are exactly *so* many *F*s, i.e. unless the question 'How many *F*s are there?' has an exact answer. If so, then principle (1) needs an extra clause on its right-hand side: there are exactly as many *F*s as *G*s iff there is a one-to-one correlation between the *F*s and the *G*s and there exists a number of *F*s and/or a number of *G*s.

This last observation is of no systematic importance. The proofs of theorems drawing consequences from the existence of a one-one map from the *F*s on to the *G*s go through whether or not this suffices for the truth of the vernacular claim 'There are exactly as many *F*s as *G*s'. (Indeed, we may in any event continue to use Frege's invented word for concepts related by such a mapping—*gleichzahlig*—along with the corresponding English invention, 'equinumerous'.) And while anything deserving the name 'theory of number' must surely deliver *some* theorems to the effect that there are exactly as many *F*s as *G*s, this condition will be met just as well by a theory that incorporates the proposed emendation of principle (1) as by one that incorporates its original, so long as it also incorporates some account of when the number of *F*s exists. (Such an account is, of course, just what principle (C) provides.) The observation does, though, provide the grounds for an *amende honorable* to the shade of Husserl. He was sceptical of principle (1)—at least as an analysis of the notion of 'exactly as many' or of 'sameness of number'—and proposed an alternative: "the simplest criterion of equality of number (*Gleichheit der Zahl*) is just that the *same* number results in counting the sets to be compared" (Husserl 1891, p. 115). In reply, Frege accused him of forgetting

> that this counting itself rests on a one-to-one correlation, namely of the numerals from 1 to *n* and the objects of the set. Each of the two sets needs to be counted. This makes the matter less simple than it is if we consider a relation that correlates the objects of the two sets without numerals as intermediaries (Frege 1894, p. 319).

This riposte has been thought to be devastating. But if not every one-to-one correlation between *F*s and *G*s validates 'There are exactly as many *F*s as *G*s', then the apparently otiose detour through the numerals does establish something crucial to the truth of this claim. Namely: that the concept *F* (and/or the concept *G*) is equinumerous with a bounded segment of some tally.

III

The arithmetical theory Θ that results from combining principles (1) and (2*) with the proposed numerical existence principle (C) does not entail that every concept has a number. Each concept *F* is equinumerous with itself, but in theory Θ that fact entails that there is a number of *F*s only given the further premiss that the Fs constitute a bounded segment of some generalised tally. Θ does not entail that every concept has this attribute, and neither does the result of adjoining to it the axiom of choice. (The axiom entails that every set may be well ordered, but there are many concepts whose extensions do not constitute sets.) More particularly, Θ also does not entail that the concept *self-identical* has this attribute, so it escapes commitment to the existence of anti-zero. Indeed, given the additional assumption that each ordinal number is an object, Θ entails that there is no such number as anti-zero. On that assumption, if there were such a thing as the number of all objects, then there would have to be such a thing as the number of all ordinal numbers. That, however, theory Θ excludes. The concept *ordinal number* is itself a generalised tally, but it cannot be equinumerous with any bounded segment of such a tally. For no bound can be placed on the ordinals themselves.

In avoiding commitment to anti-zero, Θ resembles Tennant's theory of number. But there are important differences between its existence principle (C) and Tennant's ratchet principle. The ratchet principle gives only a sufficient condition for certain cardinal numbers to exist. Principle (C), by contrast, gives a quite general necessary and sufficient condition for there to be such a thing as the number of Fs—a condition grounded in a conceptual analysis of the notion of a cardinal number (and, more particularly, of the connections between that notion and our practices of counting). If principles (1) and (2*) may be grounded similarly, then, the first of our conditions for vindicating Frege's epistemology of arithmetic is satisfied.

Does theory Θ, though, entail the Peano postulates? One strategy for showing that it does would be to fill out the derivation that Frege himself sketched in §§70–83 of *Grundlagen*, interpolating proofs that the concepts there used to define the positive natural numbers really are bounded segments of generalised tallies. Following Frege (1884 §74, p. 88), we may begin with the logical truth that the concept *not self-identical* is empty. By (C), this shows immediately that the number of non-self-identical objects exists, so that if we define '0' to stand for this number, we shall have deduced from (C) that 0 exists. Furthermore, where G is now the concept *identical with 0*, the empty relation R_0, restricted to 0, will be a strict well-ordering on G and G will be co-extensive with $G\uparrow_{R_0} 0$. Accordingly, this concept G is co-extensive with, and hence equinumerous with, a bounded segment of a generalised tally. By principle (C) again, this shows that there is such a thing as the number attaching to this concept, so that if—like Frege (1884 §77, p. 90)—we define '1' to stand for this number, we shall have deduced from (C) that 1 exists. Frege's own reasoning then shows that 1 is the number of all uniquely instantiated concepts (cfr. 1884 §78, p. 91), and also that it immediately follows 0 in the "natural sequence of numbers" (*die natürliche Zahlenreihe*; cfr. 1884 §77, p. 90). This relation, restricted to 0 and 1, is itself a well-ordering of the concept *identical with either 0 or 1*, so that this last concept too is a bounded segment of a generalised tally. This establishes the existence of 2; and so on up.

This argument could be worked up into a proof of the Peano postulates in Θ. We may, however, build on Tennant's work to establish an affirmative answer to our question more quickly. As I remarked above, he gives a free-logical derivation of the Peano postulates from principles (1) and (2*) together with the existence principles:

(A) if there are no *F*s, then the number of *F*s exists;

and

(B) if the number of *F*s exists, and there is exactly one more *G* than there are *F*s, then the number of *G*s also exists.

An analysis of his derivation shows, however, that it does not require the full power of the ratchet principle, (B). As readers of Frege will have expected, its one application in the proof comes in establishing what we might call the succession principle: that if a natural number exists then so does its successor. Now it may be shown (without appeal to the ratchet principle) that the natural number *n*—if it exists at all—is nothing other than the number of natural numbers less than *n*. It is also a logical truth that there is one more number less than or identical with *n* than there are numbers less than *n*. Accordingly, the succession principle may be derived from (B) by substituting 'natural number less than *n*' for '*F*', and 'natural number less than *n* or identical with it' for '*G*'. There is, however, something rather particular about the concepts involved in this use of the ratchet principle. Let us say that a concept is *Dedekind infinite* if it is equinumerous with one of its proper sub-concepts; and let us say that it is *Dedekind finite* otherwise. Then the concept *F* to which the ratchet principle is applied is not only Dedekind finite, but may be shown to be so on purely conceptual grounds. For if the concept 'natural number less than *n*' were Dedekind infinite, it would be equinumerous with 'natural number less than *m*', for some number *m* strictly less than *n*. And this would mean that *n* (which is the number of numbers less than *n*) would be identical with *m* (which is the number of numbers less than *m*), so that we should have *n* strictly less than *n*, and the successor of *n* less than or equal to *n*. Without invoking (B), however, Tennant shows (1987, p. 287) that this would contradict the Peano postulate that numbers with identical successors are themselves identical—a postulate that he proves without appealing to the ratchet principle (*op.cit.*, p. 295). Tennant's derivation of the Peano postulates, then, needs only the following weaker form of the ratchet principle:

(B*) if the number of *F*s exists, where *F* is a Dedekind finite concept, and there is one more *G* than there are *F*s, then the number of *G*s also exists.

So in order to show that the Peano postulates are theorems of Θ, it suffices to derive (A) and (B*) from (C).

The derivation of (A) is, as I have already remarked, immediate. But what of (B*)? Can we show that if the number of Fs exists, and F is Dedekind finite, and there is one more G than there are Fs, then the number of Gs also exists? Let us suppose that the three conjuncts in the antecedent of (B*) hold good. By principle (C) and the first conjunct, we know that F is equinumerous with a bounded segment of some generalised tally. Let us call the members of this segment the Ts, so that there is a one-one map φ from the Ts onto the Fs. Since there is exactly one more G than there are Fs, we also know that there exists an object x which is a G and for which $F \approx G \setminus \{x\}$. (Here, '$G \setminus \{x\}$' indicates the concept *a G distinct from object x*.) Let us label one such object 'a'. Then we have: (i) a is G; and (ii) $F \approx G \setminus \{a\}$, so that there exists a one-one map ψ from the Fs onto the Gs distinct from a. There are then two cases to consider. First (I), that in which each G is a T; second (II), that in which some G is not a T.

We show that case (I) contradicts the hypothesis that F is Dedekind finite. If each G is a T, then a is a T, so there will be a unique F, which we shall call b, to which object a relates under φ. We show that F is Dedekind infinite by showing that there is a one-one map from F into $F \setminus \{b\}$. Let us define the product relation

$$\chi xy \leftrightarrow \exists z\ (\psi xz \wedge \varphi zy).$$

Then χ is such a map. Since ψ maps the Fs into the Gs, and since every G is a T, ψ maps the Fs into the Ts; since φ maps the Ts into the Fs, χ maps the Fs into the Fs. For no c, however, do we have χcb. If we did, then $\exists z\ (\psi cz \wedge \varphi zb)$, and since $\forall z\ (\varphi zb \rightarrow z = a)$, we should have ψca, contradicting the claim that ψ is into $G \setminus \{a\}$. That is to say: χ maps the Fs into the Fs other than b. Moreover, the map χ is one-one because if χxy and χxw then we can find objects u and v for which $\psi xu \wedge \varphi uy$ and $\psi xv \wedge \varphi vw$. Since ψ is one-one, $u = v$, and since φ is one-one, $y = w$. The Fs, then, will be equinumerous with their images under χ, and these images will be a proper sub-concept of F, since b will not be among them. So F is Dedekind infinite.

Given the hypothesis that F is Dedekind finite we may, then, confine our attention to case (II), and suppose that there exists a G which is not a T. This time let b be such a G. We show how to construct a bounded segment of a generalised tally with which G is equinumerous. We know that there is a one-one map φ from T onto F, and that there is a one-one map ψ from F onto $G \setminus \{a\}$, so that the product map χ is one-one from T onto $G \setminus \{a\}$. We consider the relation $\chi+$ which is just like χ save that in addition $\chi{+}ba$. Since b is not a T, the $\chi+$ relation is a one-one map from $T{+}\{b\}$ on to G, where $T{+}\{b\}$ is the concept *is either a T or is identical with b*. That is to say: $G \approx T{+}\{b\}$. What is more, $T{+}\{b\}$ is a bounded segment of a generalised tally. We are given that T

has this attribute, so that the Ts may be well ordered by a relation R which provides an upper bound for them. We extend R to an order $R+$ on $T+\{b\}$ by adding b "at the front", i.e. by picking a d which is least in R and defining $R+$ to be just like R save that in addition $R{+}bd$. $R+$ inherits its transitivity and connectedness from R. Moreover, since b is not a T, the fact that R is irreflexive on T entails that $R+$ is irreflexive on $T+\{b\}$. Furthermore, any non-empty subconcept Y of $T+\{b\}$ will have an $R+$-least object falling under it. (If b falls under Y, this object will be b itself. If not, then Y will be a non-empty subconcept of T, and the $R+$-least object in Y will be the R-least object, which is guaranteed to exist since T is well ordered by R.) Finally, the upper bound that R provides for T also serves as an $R+$-upper bound for $T+\{b\}$; for $R+$ differs from R only at the front. This completes the proof that $T+\{b\}$ is a bounded segment of a generalised tally, and since G is equinumerous with it, (C) entails that there exists such a thing as the number of Gs. And this in turn completes the derivation of principle (B*) from principle (C).

The derivation shows the significance of the difference between principles (B) and (B*). The problem is not that principle (B)—the strong version of the ratchet principle—is false in case (I). On the contrary; if each G is a T, then there will be a one-one map from the Gs into the Fs—namely, the restriction of φ to Gs. Since there is also a one-one map from the Fs into the Gs—viz. ψ—the Schröder-Bernstein theorem shows that Fs and Gs are equinumerous, so that if the number of Fs exists then so certainly does the number of Gs. But while there is no serious doubt about the truth of the Schröder-Bernstein theorem, its proof uses a definition by recursion, and nobody concerned to vindicate Frege's epistemology of arithmetic can appeal to it in deriving the Peano postulates. For the question would then arise, whether the legitimacy of definition by recursion rests upon some "intuitive", or at least non-conceptual, element. Whatever the answer to that question, however, the derivation just given of (B*) from (C) may be formalised as a strictly logical proof.

In drawing attention to this difference, I do not mean to exclude the possibility that conceptual analysis of the notion of cardinal number might establish the strong ratchet principle (B), or that it might show more directly the analyticity of the weak ratchet principle (B*). (Indeed, I cannot consistently exclude the second possibility; for (B*) is a logical consequence of a thesis that I claim to be knowable through conceptual analysis, viz. (C).) All the same, even if one or both of these possibilities were to be realised, the existence principle (C) would retain its interest. Even in its strong form as (B) rather than (B*), the ratchet principle tells us nothing about the existence of any infinite cardinals. Principle (C), by contrast, tells us quite generally that the number of Fs exists whenever the Fs are equinumerous with a bounded segment of some generalised tally. The principle may be applied, then, to establish the existence of transfinite cardinals. Applying the principle to establish the existence of an infinite number of Fs will, though, involve finding a well-orderable concept G with which the Fs may be shown to be equinumerous, and in general it will not be possible to show on purely logico-conceptual grounds that such a G exists. In one special case, however, this would appear to be possible. The Peano postulates themselves entail that the natural numbers are strictly well ordered by the relation $<$. Let us suppose, then, that we can establish on a logic-conceptual basis the existence of an object α that is not a natural number. (An example of an object for which this supposition is plausible might be the two-membered set {0,1}.) Then we can extend the relation $<$ to another well-ordering R on the natural numbers together with α by stipulating that every natural number R-relates to α—i.e. by treating α as a maximal element under the extended ordering. This shows that the natural numbers are equinumerous with a bounded segment of a generalised tally, so that principle (C) would entail the existence of the number $\aleph_0$ of natural numbers. On our supposition, then, the existence of $\aleph_0$—and in fact its elementary arithmetic—could be established on a logico-conceptual basis. This is a satisfactorily Fregean result. The long section Iota of Part II of *Grundgesetze* is devoted to proving theorems about the *Anzahl Endlos*, defined as the number of finite cardinals (1893 §122, p. 150). Our analysis suggests that Frege was right to think that the arithmetic of $\aleph_0$ or *Endlos* could be developed on the basis of logic and definitions alone as easily as could the arithmetic of the natural numbers. The arithmetic of $\aleph_0$, however, is only the first chapter in the long saga of transfinite arithmetic, and the analysis also suggests that this may be the only chapter that can be developed on a purely conceptual basis. For it is hard to see how to derive even the existence of $\aleph_1$ from (C) without making assumptions about well-orderability which (although they may very well be true) go beyond anything that could be grounded in logic and definitions alone. But while this will disappoint those who are ambitious to place the whole of transfinite arithmetic on a logical basis, there is a coherent logicist project which includes no such ambition. For it might be—to paraphrase Kronecker—that logic "makes" the natural numbers, and their own cardinality $\aleph_0$, whereas some other source of knowledge is required to ground the existence of higher cardinals.

The derivation of the Peano postulates that has just been sketched rests on the Fregean assumption that expressions meaning 'the number of so-and-sos' can be construed as singular terms—i.e., as expressions which at least purport to designate particular, re-identifiable objects. But while this assumption has been subject to serious challenge,[18] the use of unrestricted quantification suggests an alternative strategy for proving the postulates which may be available even to one who rejects the assumption. For suppose that a theorist instead pursues what Dummett has called the "radical adjectival strategy", whereby "equations and other arithmetical statements in which numerals apparently figure as singular terms are to be explained...by *transforming* them into sentences in which number-words occur only adjectivally" (Dummett 1991, p. 99). For example, the equation '5 + 7 = 12' might be explained as a condensed version of the following claim, in which the conditional is taken materially:

> Whatever concepts F and G might be, if there are precisely five things that are F and precisely seven things that are G, and nothing is both F and G, then there are precisely twelve things that are either F or G.

The problem in combining this sort of analysis of purely arithmetical statements with any variety of logicism has always been that the analysis requires an infinite domain of objects if manifest arithmetical falsehoods are not to be transformed into truths. If, for example, there were only eleven objects, then the matrix antecedent of the above claim would be false for any concepts F and G, so that a universally quantified material conditional with this antecedent would be true whatever its consequent might be. This would mean that, were there only eleven objects, the claims into which the equations '$5 + 7 = 11$' and '$5 + 7 = 13$' are transformed would be true. Needless to say, those attracted by versions of the radical adjectival strategy have explored a number of solutions to this problem. But it is noteworthy that the natural presentation of the logic of unrestricted quantification renders it logically impossible that there should be (speaking unrestrictedly) only finitely many objects, and it does so without invoking the assumption that numbers are objects. For where σ_n is the first-order formula which expresses 'There are (unrestrictedly) at least n objects',[19] each member of the sequence $\sigma_1, \sigma_2, \ldots$ is a logical truth even in the first-order logic of unrestricted quantification.

At least, this is so if the standard model-theoretic definition of logical truth is extended to cover unrestricted quantification in the obvious way. For let A be a well formed formula containing only variables, quantifiers and predicate letters. According to the standard definition, when its variables are understood to range restrictedly, A is logically true if it is true no matter which subset D of the model theory's ontology U provides assignments for its variables, and no matter which subsets of D^n are then assigned to its n-place non-logical predicate letters. It is then natural to say that when its variables range *un*restrictedly, A is logically true if it is true when assignments to its variables may be made from the entire model-theoretic ontology U, no matter which subsets of U^n are assigned to its n-place logical predicate letters. Now each formula σ_n contains no non-logical predicate letters. Accordingly, each such formula will be logically true if it is true when assignments to its variables may be made from the whole of U. And each σ_n is then logically true, for the model theoretic ontology U must be infinite. Or at least, it must be infinite if (for example) Gödel's completeness theorem for the first-order calculus is to be true. That theorem entails that each consistent formula is satisfiable. But while (for example) the formula

$$\forall x \exists y\, Sxy \wedge \forall x\, \neg Sxx \wedge \forall x \forall y \forall z\, (Sxy \wedge Syz \rightarrow Sxz)$$

is consistent, it cannot be satisfied when only finitely many objects are available to be assigned to its objectual variables. This point is, indeed, implicit in the standard Henkin proof of the completeness theorem. Henkin's "term model" exploits the ontological resources of the model theory's syntactic component; and that syntactic component posits an infinite stock of variables, which constitute the domain of quantification for the term model. Whether the validity of each σ_n in the logic of unrestricted quantification gives a logicist who pursues the radical adjectival strategy quite what he needs to dispel the bugbear of a shortage of objects depends upon delicate questions concerning the nature of his logicism. But it shows how there may be varieties of logicism which do not depend on the Fregean thesis that numbers are objects.

Even if we assume that thesis, I do not imagine that the derivation I have given of the Peano postulates completes the vindication of Frege's mathematical epistemology even for finite arithmetic. In order to complete this, it would further need to be shown how the ancillary principles used in the derivation—and (N2*) in particular—may be ground out of the mills of analysis. We would also need some assurance that conceptual analysis, when applied to arithmetical notions, really is a means of gaining knowledge. We would need, in other words, to exclude the possibility that analysis simply brings to the surface falsehoods that are latent in our use of arithmetical terms. I venture to conclude, however, with two comparative claims. By virtue of its more general numerical existence principle, the derivation of the Peano postulates within Θ provides a better basis for any attempt to vindicate Frege's epistemology of arithmetic than does Tennant's derivation of them from the ratchet principle. And their derivation within Θ also improves for this purpose on their derivation from Hume's principle. For Θ's axioms, unlike Hume's principle, are not committed to the existence of anti-zero.[20]

Notes

[1]See e.g. Wright 1983, chap. 4.

[2]See here Heck 1993.

[3]Boolos 1990, p. 274. Cfr. Boolos 1997, p. 260, where he calls this worry about Hume's principle "perhaps the most serious of all, although one that may at first appear to be dismissible as silly or trivial".

[4]The so-called "numeration theorem" in ZF—a consequence of the axiom of choice—asserts the converse: any set is equinumerous with some ordinal.

[5]Frege 1903 §157, p. 155. Compare Frege 1893 §41, p. 58, where he warns us that "it will later turn out to be necessary to distinguish the cardinal number zero from the real number zero". It is worth noting in passing that, when combined with another of Frege's doctrines—viz. that "affirmation of existence is nothing but denial of the number zero" (Frege 1884, §53, p. 65)—this entails an ambiguity in the expression 'there is'. This is because the zero that is denied by saying 'there is' varies from case to case. When I affirm 'There is a golden mountain', I am denying the natural number. I am excluding the natural number zero as an answer to the question 'How many golden mountains are there?' But in saying 'There is gold in Fafner's cave' I am denying the real number. I am excluding the real number zero as an answer to the very different question 'How much gold is there in Fafner's cave?'

[6]This may, indeed, be Wittgenstein's ground for deeming it "nonsensical to speak of the *total number of objects*" at *Tractatus* 4.1272. A sentence containing such a phrase will be a "nonsensical pseudo-proposition", borne out of misconstruing the grammatical common noun 'object' "as a

proper concept-word" (*als eigentliche Begriffswort*) (*ibid.*). The fact (if it is one) that disjoining two count nouns does not always produce another such expression explains why treating 'object' as a concept-word is indeed a mistake.

[7]See Burgess 1984 and Boolos 1987, pp. 9–10.

[8]If, indeed, it is apposite to talk here about an inferential "move" from one content to another. In *Grundlagen*, although not in *Grundgesetze*, Frege writes that the content of a judgement of equinumerosity "lets itself be taken (*sich lassen auffassen*) as an identity each of whose sides is a number" (1884 §63, p. 74). Similarly, the two sides of instances of Dummett's principle (2) are said to "carve up" a single content in different ways (§64, p. 75). For an attempt to make something out of this, see Hale 1997.

[9]In fact, it would be better to say that we need such a logic unless we are prepared to "Russell away" formulae involving numerical terms in favour of descriptive formulae involving an unequal-levelled numbering relation 'n num F', so that '$\varphi(Nx{:}Fx)$' is replaced by the explicitly quantified formula '$\exists n[\forall m\ (m \text{ num } F \leftrightarrow m = n) \wedge \varphi n]$'. See Boolos 1996. For present purposes, however, nothing turns on the difference between these two approaches. So, for ease of comparison with his work, I shall follow Tennant in considering how arithmetic might be axiomatized within a free logic.

[10]For the derivation see Tennant 1987, pp. 275–300. In fact, I simplify somewhat here. While Boolos and Wright formulate their arithmetical theories in a language whose only non-logical expression is the functor 'N', Tennant formulates his in a language with three non-logical primitives: 'N' again; the singular term '*0*' (intended to mean 'the natural number zero'); and the function sign '$s(\xi)$' (intended to mean 'successor'). For this reason, his proofs of the Peano postulates rest upon three existence principles:

1. If there are no Fs then $NxFx = 0$
2. If $NxFx$ exists and there is exactly one more G than there are Fs then $NxGx$ exists
3. If $t = Nx{:}\ Fx$ and there is exactly one more G than there are Fs, then $Nx{:}Gx = s(t)$.

Now there are, I think, good reasons for preferring an approach such as Tennant's whereby '*0*' and '$s(\xi)$' are treated as primitives. (See Rumfitt 1999.) Those reasons, however, lie at some distance from present concerns, so for ease of comparison with Frege, Boolos and Wright, I shall focus on the existence axioms formulated in the text, which are cast in their more restricted language. No substantive issues are involved, for my axioms (A) and (B) entail Tennant's axioms (1) to (3) when combined with explicit definitions which will be needed in any language whose only arithmetical primitive is 'N'. For axiom (B) is precisely Tennant's axiom (2), and it yields his axiom (3) when supplemented with Frege's definition of '$s(\xi)$', a definition which both Boolos and Wright accept. Similarly, axiom (A) yields Tennant's axiom (1) when supplemented with Frege's definition of '*0*', which they also accept.

[11]I shall assume for the sake of argument that principles (1) and (2*) are so knowable. But see the last two paragraphs of the present section.

[12]It should be noted, however, that 'How many?' questions admit of perfectly correct *inexact* answers. I explore some consequences of this fact—wholly congenial, I think, to the position to be defended here—in Rumfitt 2001.

[13]Cfr. Frege 1893 §37, p. 55. If R is functional then there is at most one object to which any given object stands in the relation R; but it is not required that there should be at least one. So the "function" that maps each object to its R-image need not be total, and the relation of immediately preceding, when restricted to the numbers 1 to n, is still functional.

[14]Where R is any relation, its strong ancestral is the relation R^* for which

$$R^*xy \leftrightarrow \forall F\ [\forall z\ (Rxz \rightarrow Fz) \wedge \forall z \forall w\ ((Fz \wedge Rzw) \rightarrow Fw) \rightarrow Fy].$$

See e.g. Frege 1893, §45, pp. 59–60. R's weak ancestral $R^{*=}$ is the disjunction of its strong ancestral with the relation of identity; see Frege 1893 §46, p. 60.

[15]Frege 1893 §108, p. 137. Heck (1998, pp. 449–53) also comments upon this passage, but with a view to showing that "Frege's characterisation of finitude is reasonably close to Zermelo's" (p. 453).

[16]Frege in effect proved this in *Grundgesetze*. By Theorem 275 of that work, any strong ancestral is a transitive relation; by Theorem 243 the strong ancestral of a functional relation is connected. So, given also that it is irreflexive (condition (iii)), the strong ancestral of a relation meeting conditions (ii) and (iii) will be a linear ordering. Finally, theorem 359 of *Grundgesetze* shows that any relation meeting conditions (i) to (iii) will be such that any non-empty subset of its relata possesses an R^*-minimal member.

[17]These differences remain even when, as lawyers might put it, questions of the two kinds are posed in relation to the same facts. Dummett remarks that

> if Frege had paid more attention to Cantor's work, he would have understood what it revealed, that the notion of an ordinal number is more fundamental than that of a cardinal number. This is true even in the finite case; after all, when we count the strokes of a clock, we are assigning an ordinal number rather than a cardinal. If Frege had understood this, he would therefore have characterised the natural numbers as finite ordinals rather than as finite cardinals (Dummett 1991, p. 293).

But the questions 'Which stroke was that?' and 'How many strokes have we heard?' are different, and demand different sorts of answer ('the eleventh', for example, *versus* 'eleven'), even though somebody who attentively counts the strokes will thereby be in a position to answer both.

Note, too, that the fact (if it is one) that well ordered relations are fundamental to cardinal arithmetic does not show that ordinal numbers are. For it is a further step—and a problematical one—to abstract an order-type from well ordered relations. In §86 of *Grundlagen*, Frege claims to be able to anticipate how Cantor's notion of an order-type could be made precise (1884, p.98). As Harold Hodes has observed (1984, p.138), however, the obvious attempt to generate order-types through Fregean abstraction on well ordered relations falls foul of Burali-Forti's paradox.

[18]See again Hodes 1984.

[19]Thus σ_1 might be '$\exists x_1\ (x_1 = x_1)$', σ_2 might be '$\exists x_1\ \exists x_2 \neg\ (x_1 = x_2)$',..., in which the variables 'x_1', 'x_2',... are all understood to range unrestrictedly.

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