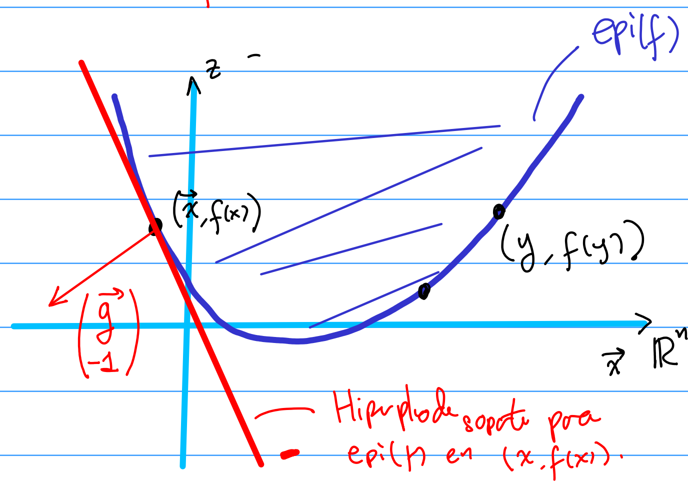
Métodos de subgradientes:

Sea $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ convexa

Def: El subdiferencial de f en $\vec{a} \in \mathbb{R}^n$ es

$$\partial f(\vec{a}) = \{ g \in \mathbb{R}^n : \forall y \in \mathrm{dom}(f) \quad f(y) \geq f(a) + g^t(y-a) \} \subseteq \mathbb{R}^n$$

Geometría de $\partial f(\vec{a})$:



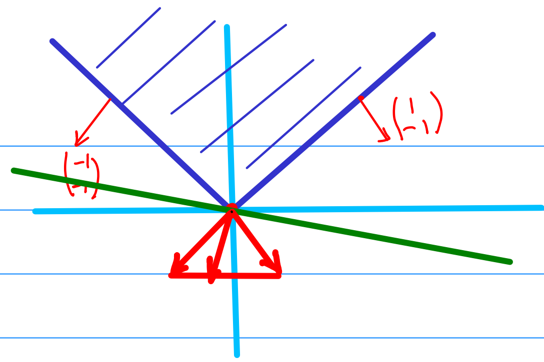
Lema: $\begin{pmatrix} \vec{g} \\ -1 \end{pmatrix} \in \mathbb{R}^{n+1}$ es normal de un hip de soporte en $(\vec{x}, f(\vec{x}))$

$\Longleftrightarrow g \in \partial f(\vec{x})$

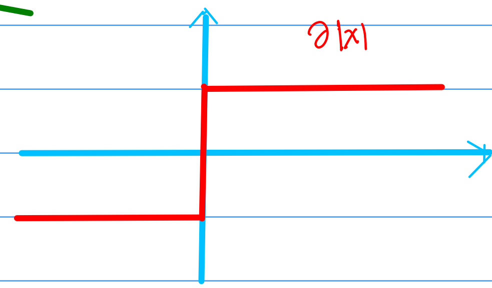
$$\begin{pmatrix} g \\ -1 \end{pmatrix} \cdot \begin{pmatrix} x \\ f(x) \end{pmatrix} \geq \begin{pmatrix} g \\ -1 \end{pmatrix} \cdot \begin{pmatrix} y \\ f(y) \end{pmatrix}$$

$$g^t x - f(x) \geq g^t y - f(y)$$

$$[f(y) \geq f(x) + g^t(y-x)]$$



$f(x) = |x|$



$$\partial |x| (a) = \begin{cases} -1, & a<0 \\ [-1,1], & a=0 \\ 1, & a>0 \end{cases}$$

Ejemplo

Obs: (1) $\partial f(a)$ es convexo

(2) Si f es convexa y $a \in \text{int}(\text{dom}(f))$

$\partial f(a) \neq \emptyset$ (Hahn-Banach)

(3) $h(x) = \max(f_1(x), f_2(x))$

$\partial h(a) = \text{Conv}(\partial f_1(a), \partial f_2(a))$

(*) (4) Si f es convexa y diferenciable y $\vec{a} \in \text{int}(\text{dom}(f))$ entonces

$$\partial f(\vec{a}) = \{\nabla f(\vec{a})\}.$$

Lema: x^* es un mínimo GLOBAL de f ssi $\partial f(x^*) \ni \vec{0}$.

Dem: x^* es mínimo global de f $\iff$

$\forall y \in \text{dom}(f) \quad (f(y) \geq f(x^*))$

$\iff \quad f(y) \geq f(x^*) + \vec{0}^t(y - x^*)$

$\iff \vec{0} \in \partial f(x^*)$

Corolario: Sea $f: \mathbb{R}^n \to \mathbb{R}$ convexa y C^1 (dif con cont.)

x^* es mínimo global $\iff \nabla f(x^*) = 0$.

Suponga $f: \mathbb{R}^n \to \mathbb{R}$ convexa y de Lipschitz

$\big(\|f(v)-f(u)\| \le G \|v-u\|$ f no varía demasiado rápido$\big)$

y que dado $\vec{a} \in \mathbb{R}^n$ podemos pedir $g \in \partial f(\vec{a})$

Algoritmo: [Método del subgradiente]

$$\begin{cases} x^{(0)} \text{ arbitrario} \\ x^{(k+1)} := x^{(k)} - \underbrace{\alpha_k}_{\text{tamaño del paso}} g_k \end{cases} \quad \text{con } \alpha_k \in \mathbb{R}, \quad g_k \in \partial f(x^{(k)})$$

$$\left[f_k := \min_{0 \le j \le k} f(x^{(j)}) \right]$$

Teorema: Si $\sum_{k=0}^{\infty} \alpha_k = \infty$ y $\sum_{k=0}^{\infty} \alpha_k^2 < \infty$ $\quad \alpha_k \ge 0$

entonces $f_k \xrightarrow[k \to \infty]{} f_*$ ← mínimo global de f.

Dem: Sea x^* un minimizador de f (i.e. $f(x^*) = f_*$)

$$\|x^{(k+1)} - x^*\|_2^2 = \|x^{(k)} - \alpha_k g_k - x^*\|_2^2 = \|(x^{(k)} - x^*) - \alpha_k g_k\|_2^2$$

$$= \|x^{(k)} - x^*\|_2^2 + \alpha_k^2 \|g_k\|_2^2 + 2\alpha_k \underbrace{g_k}_{\in \partial f(x^{(k)})} \cdot (x^* - x^{(k)})$$

$$f(x^*) \ge f(x^{(k)}) + g_k \cdot (x^* - x^{(k)})$$

$$\left[\|x^{(k+1)} - x^*\|_2^2 \le \|x^{(k)} - x^*\|_2^2 + \alpha_k^2 \underbrace{\|g_k\|_2^2}_{G^2} + 2\alpha_k \big(f(x^*) - f(x^{(k)}) \big) \right] (*)$$

Lipschitz:

$$G\|y - x^{(k)}\| \ge f(y) - f(x^{(k)}) \ge g_k \cdot (y - x^{(k)}) \qquad \forall y$$

Si $y = x^{(k)} + \vec{u}$, $\|u\|_2 = 1$

$$G \underbrace{\|\vec{u}\|}_{1} \ge g_k \cdot \vec{u} \underset{\text{tomando } \sup \vec{u}}{\Longrightarrow} G \ge \|g_k\|_2$$

$\forall k$

$$\|x^{(k+1)} - x^*\|_2^2 \le \|x^{(0)} - x^*\|_2^2 + \sum_{j=0}^{k+1} \alpha_j^2 G^2 + 2 \sum_{j=0}^{k+1} \alpha_j \big(f(x^*) - f(x^{(j)}) \big)$$

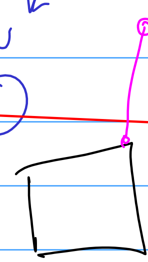
$\forall k$

$$0 \le \|x^{(0)}-x^*\|_2^2 + \sum_{j=0}^{k+1} \alpha_j^2 G^2 + 2\sum_{j=0}^{k+1}\alpha_j \left(f_* - f(x^{(j)})\right)$$

$$\sum_{j=0}^{k+1}\alpha_j \left(f(x^{(j)}) - f_*\right) \le \frac{\|x^{(0)}-x^*\|_2^2 + \sum_{j=0}^{k+1}\alpha_j^2 G^2}{2}$$

$$\sum_{j=0}^{k+1}\alpha_j \left(f(x^{(j)}) - f_*\right) \geq \left[\min_{0\le j\le k+1} \left[f(x^{(j)}) - f_*\right]\right]\left(\sum_{j=0}^{k+1}\alpha_j\right)$$

$$\boxed{f_{k+1} - f_* \le \frac{\|x^{(0)}-x^*\|_2^2 + \sum_{j=0}^{k+1}\alpha_j^2 G^2}{2\sum_{j=0}^{k+1}\alpha_j} \;\xrightarrow{k\to\infty}\; 0}$$



Subgradiente proyectado

Sea $f: \mathbb{R}^n \longrightarrow \mathbb{R}$ convexa, Lipschitz y $C \subseteq \mathbb{R}^n$ un convexo del que conocemos una proyección $P_C: \mathbb{R}^n \longrightarrow C$

$$z \longmapsto P_C(z) = \arg\min_{y\in C}\|y-z\|$$

Problema $\min f(x)$ s.a. $x \in C$

$x^{(0)}$ arbitrario en C

$z^{(0)} = x^{(0)}$

$$\begin{cases} z^{(k+1)} := x^{(k)} - \alpha_k g_k, & g_k \in \partial f(x^{(k)}) \\ x^{(k+1)} = P_C(z^{(k)}) \end{cases}$$

$$f_k = \min_{0 \leq j \leq k} f(x^{(k)})$$

Si Problema tiene algún mínimo $\in C$ entonces

Teorema: Si $\sum \alpha_k = \infty$ $\sum \alpha_k^2 < \infty$ $\alpha_k \geq 0$

entonces el método converge a algún mínimo global $x^* \in C$.