Clase anterior, (1) Aclarar la desigualdad
(2) Unicidad de la extensión ?

Clase de hoy: (3) Convexos K con $\mathrm{int}(K) \neq \emptyset$ ~ ?

$\Updownarrow$

Bolas unitarias de norma ~ $\|\cdot\|$

(1) $M \subseteq V$, $y \notin M$ $\quad [f(x) \leq \|f\|_* \|x\|]$

$f: M \longrightarrow \mathbb{R}$ $\quad \|f\|_* < \infty$

Cómo extender f en $\langle M, y\rangle$ con $\|g\|_* \leq \|f\|_*$

$g: \langle M, y \rangle \longrightarrow \mathbb{R}$

$x = m + \alpha y$

Queremos:

$$g(x) \leq \|f\|_* \|x\| \quad (\|g\|_* \leq \|f\|_*)$$

$$g(m+\alpha y) = g(m) + \alpha g(y) = f(m) + \alpha g(y)$$

Queremos $\left\{ \left[f(m) + \alpha \boxed{g(y)}^{?} \leq \|f\|_* \|m + \alpha y\| \right] \right\}$

Si $\alpha > 0$:

$$g(y) \leq \frac{\|f\|_* \|m+\alpha y\| - f(m)}{\alpha}$$

$$g(y) \leq \|f\|_* \left\| \frac{m}{\alpha} + y \right\| - f\left(\frac{m}{\alpha}\right)$$

Si $\alpha < 0$:

$$g(y) \geq \frac{\|f\|_* \|m+\alpha y\| - f(m)}{\alpha}$$

$$g(y) \geq f\left(\frac{m}{-\alpha}\right) - \|f\|_* \left\| -y - \frac{m}{\alpha} \right\|$$

$\forall m_1, m_2$

$$f(m_2) - \|f\|_* \|m_2 - y\| \leq g(y) \leq \|f\|_* \|m_1 + y\| - f(m_1)$$

$$\Downarrow$$

$$f(m_1) + f(m_2) \leq \|f\|_* \left(\|m_1 + y\| + \|m_2 - y\|\right)$$

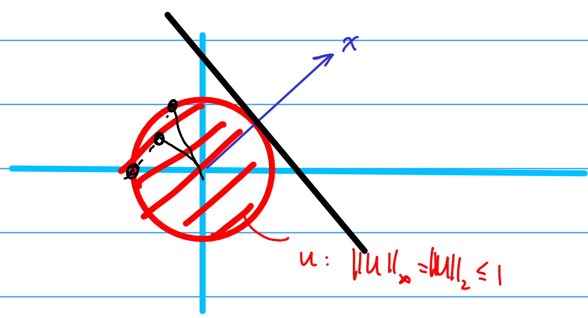
$$\vee\!/$$

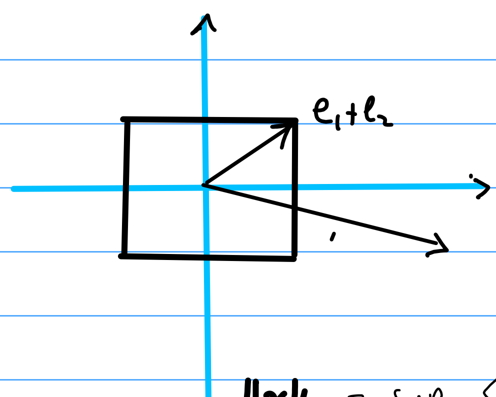
$$f(m_1) + f(m_2) = f(m_1 + m_2) \leq \|f\|_* \|m_1 + m_2\|$$

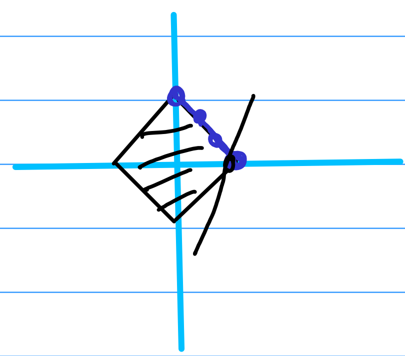
Unidad ?

$$\|x\| = \sup_{u : \|u\|_* \leq 1} \langle u, x \rangle$$

Dado x, cuántos u's : $\|u\|_* \leq 1$ satisfacen $\langle u, x \rangle = \|x\|$?





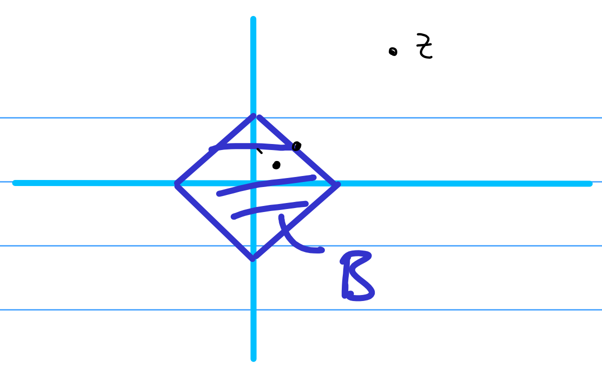


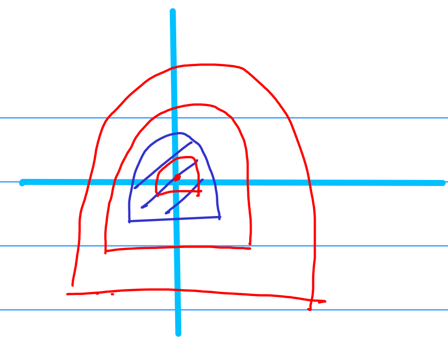
$$\|x\|_\infty = \sup_{y : \|y\|_1 \leq 1} \langle x, y \rangle$$

$$\langle \alpha V_1 + \beta V_2, e_1 + e_2 \rangle \quad \alpha + \beta = 1$$

$$\|e_1 + e_2\|_\infty = 1$$

Positivamente homogéneo : $p(\lambda \vec{x}) = \lambda p(\vec{x}) \quad \forall \lambda > 0.$





$$\|z\| = \inf\{\lambda>0 : \lambda B \ni z\}$$
$$= \inf\{\lambda>0 : \tfrac{z}{\lambda} \in B\}$$

V e.v. normado $/ \mathbb{R}$

Definición: Si $K \subseteq V$ es convexo y $0 \in \mathrm{int}(K)$

definimos

$$\boxed{p_K(x) = \inf\{\lambda>0 : \lambda K \ni x\}}$$

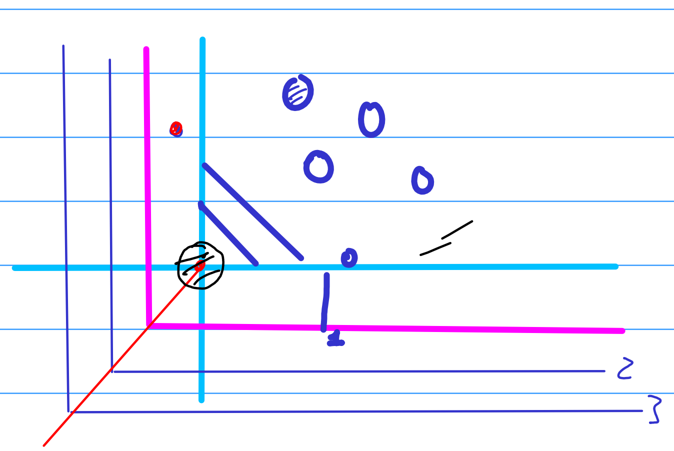
Teorema:

✓(1) $\infty > p_K(x) \geq 0$
✓(2) p_K es convexo
✓(3) p_K es positivamente homogéneo

$\}$ p_K es una seminorma

(4) p_K es continua
(5) $\{x \in V : p_K(x) \leq 1\} = \overline{K}$
$\{x \in V : p_K(x) < 1\} = \mathrm{int}(K)$

Ejemplo:



(1) $p_K(x) < \infty$ porque $0 \in$ int

(2) $\left[\begin{array}{l} p_K(x_1) = \beta_1 \\ p_K(x_2) = \beta_2 \end{array}\right] \Rightarrow p_K\left(\frac{x_1}{\beta_1}\right) = 1, \quad p_K\left(\frac{x_2}{\beta_2}\right) = 1$

$aK + bK \subseteq (a+b)K$

$x_1 \in (\beta_1 + \varepsilon) K$

$x_2 \in (\beta_2 + \varepsilon) K$

$\lambda x_1 + (1-\lambda) x_2 \in \left[\lambda(\beta_1+\varepsilon) K + (1-\lambda)(\beta_2+\varepsilon) K\right]$

convexidad de K

$\subseteq \lambda \beta_1 K + \lambda \varepsilon K + (1-\lambda)\beta_2 K + (1-\lambda)\varepsilon K$

$\subseteq \left[\lambda \beta_1 K + (1-\lambda)\beta_2 K\right] + \varepsilon K$

$\lambda \frac{x_1}{\beta_1} + (1-\lambda) \frac{x_2}{\beta_2}$

$\frac{x_1}{\beta_1} \in (1+\varepsilon) K$

$\frac{x_2}{\beta_2} \in (1+\varepsilon) K$

$\lambda \frac{x_1}{\beta_1} + (1-\lambda) \frac{x_2}{\beta_2} \in \lambda (1+\varepsilon) K + (1-\lambda)(1+\varepsilon) K$

$\subseteq K + \varepsilon K$

$\left[p\left(\lambda \frac{x_1}{\beta_1} + (1-\lambda) \frac{x_2}{\beta_2}\right) \leq 1 \right] \qquad \lambda = \frac{\beta_1}{\beta_1 + \beta_2}$

$1 - \lambda = \frac{\beta_2}{\beta_1 + \beta_2}$

$p\left(\frac{x_1}{\beta_1+\beta_2} + \frac{x_2}{\beta_1+\beta_2}\right) \leq 1$

$\left[p(x_1 + x_2) \leq \beta_1 + \beta_2 = p(x_1) + p(x_2) \right]$

Asumimos que p es pos homog y $\Rightarrow$ Convexa

concluimos $p(x_1 + x_2) \leq p(x_1) + p(x_2)$

$p_K(\alpha\vec{x}) = \alpha\, p_K(\vec{x})\ ,\quad \alpha > 0$

Sea $\lambda > 0$: $\quad \vec{x} \in \lambda K$

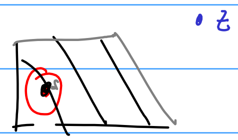
$\Rightarrow \quad \alpha\lambda > 0$ y $\quad \alpha\vec{x} \in (\alpha\lambda)K$

$\Rightarrow \quad \alpha\lambda \in \{\eta > 0,\ \eta K \ni \alpha\vec{x}\}$

$\alpha\, p_K(\vec{x}) \geq p_K(\alpha\vec{x})$

$\leq$

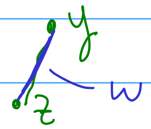
(4)



$p_K(z) \leq p_B(z) = \frac{1}{\delta}\|z\|$

Luego $p_K(z)$ es continua en 0

$|p_K(z) - p_K(0)| \leq \frac{1}{\delta}\|z\|$ ✓



$p_K(z) = p_K(z - y + y) \leq p_K(y) + p_K(z-y)$

$|p_K(z) - p_K(y)| \leq \boxed{|p_K(z-y)|} \leq \frac{1}{\delta}\|z - y\|$

Ejercicio: (a) Chequear $p_K(x) = 0$ en prueba anterior

(b) Demostrar que si p es pos homogeneo

p es convexa $\iff p(x+y) \leq p(x) + p(y)$.