Hoy: Potencias simétricas y exteriores de representaciones

Suponga que (V, ρ_V) es una rep. de G

Def: Sea $d \in \mathbb{N}$, $\left(\mathrm{Sym}^d(V), \rho_{\mathrm{Sym}^d(V)}\right)$

Si $T : V \longrightarrow W$ definiremos $\mathrm{Sym}^d(T) : \mathrm{Sym}^d(V) \longrightarrow \mathrm{Sym}^d(W)$ usando la propiedad univ del producto simétrico

$$(\vec{v}_1, \dots, \vec{v}_d) \xmapsto{\text{d-lineal y simétrica}} T(\vec{v}_1) \cdot \ldots \cdot T(\vec{v}_d) := \psi\left(T(\vec{v}_1), \dots, T(\vec{v}_d)\right)$$

$$\begin{tikzcd}
\underbrace{V \times \cdots \times V}_{d\text{-copias}} \arrow[rr] \arrow[d, "\psi"'] & & \mathrm{Sym}^d(W) \\
\mathrm{Sym}^d(V) \arrow[urr, dashed, "\exists!\, \varphi"'] & &
\end{tikzcd}$$

$\exists!\, \varphi =: \mathrm{Sym}^d(T)$ lineal

$\varphi(v_1 \cdot \ldots \cdot v_d) = T(v_1) \cdot \ldots \cdot T(v_d)$ ✓

$$\rho_{\mathrm{Sym}^d(V)}(g) := \mathrm{Sym}^d\left(\rho_V(g)\right)$$

Ejemplo: $V = \langle e_1, e_2 \rangle$, $B_V = \{e_1, e_2\}$ $\quad T : V \longrightarrow V$

$$[T]_{B_V} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$d = 2$, calculemos $\mathrm{Sym}^2(T) : \mathrm{Sym}^2(V) \longrightarrow \mathrm{Sym}^2(V)$

Sol:

$\mathrm{Sym}^2(V) = \langle e_1^2, e_1 e_2, e_2^2 \rangle$ $\quad B_S = \{e_1^2, e_1 e_2, e_2^2\}$

$$\mathrm{Sym}^2(T)(e_1^2) = (T(e_1))(T(e_1)) = (a e_1 + c e_2)^2 = a^2 e_1^2 + 2ac\, e_1 e_2 + c^2 e_2^2$$

$$\mathrm{Sym}^2(T)(e_1 e_2) = T(e_1) T(e_2) = (a e_1 + c e_2) \cdot (b e_1 + d e_2) = ab\, e_1^2 + (ad + cb) e_1 e_2 + cd\, e_2^2$$

$$\begin{array}{c|ccc|}
 & e_1^2 & e_1e_2 & e_2^2 \\
\hline
e_1^2 & a^2 & ab & b^2 \\
e_1e_2 & 2ac & ad+bc & 2bd \\
e_2^2 & c^2 & cd & d^2
\end{array} = \left[\mathrm{Sym}^2(T)\right] \longleftarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Ejercicio: Calcule $\left[\mathrm{Sym}^3(T)\right]_{B_{\mathrm{Sym}^3(V)}}$

Ejercicio: $\mathrm{Sym}^d(V)^* \cong \mathrm{Sym}^d(V^*)$ como reps.

Obs: Si $V = \langle e_1, e_2, e_3 \rangle$, $V^* = \langle x_1, x_2, x_3 \rangle$

$x_i(e_j) = \delta_{ij}$

$\mathrm{Sym}^d(V^*) = \mathbb{C}[x_1, x_2, x_3]_d \leftarrow$ polinomios homogéneos de grado d

$$\underbrace{\mathbb{C}[x_1,x_2,x_3]}_{\text{contragrediente}} \underset{\text{iso de reps.}}{=} \underbrace{\bigoplus_{k=0}^{\infty} \mathrm{Sym}^k(V^*)}_{\text{suma directa de las representaciones}}$$

Potencias exteriores:

Si V es un espacio vectorial y $k \in \mathbb{N}$ queremos definir $(\Lambda^k V, \wedge)$ que linealicen las operaciones k-lineales alternantes. Más precisamente, queremos:

$$\begin{tikzcd}
\overbrace{V \times \cdots \times V}^{k\text{-veces}} \arrow[r, "m"] \arrow[d, "\wedge"'] & S \\
\Lambda^k V \arrow[ur, dashed, "\exists!\, \varphi"']
\end{tikzcd}$$

$\forall\, m$ — k-lineal y alternante

espacio vect cualquiera

$\forall \sigma \in S_k$: $m(v_{\sigma(1)}, \dots, v_{\sigma(k)}) = \underset{\{-1,1\}}{\mathrm{sgn}(\sigma)}\, m(v_1, \dots, v_k)$

k-lineal alternante

Lineal

Construcción:

Construcción: $V = \langle e_1, \dots, e_n \rangle \quad k \in \mathbb{N}$

$$\Lambda^k V = \left\langle e_{i_1} \wedge \dots \wedge e_{i_k} : 1 \leq i_1 < i_2 < \dots < i_k \leq n \right\rangle$$

$\binom{n}{k}$ símbolos

$$\wedge : \underbrace{V \times \dots \times V}_{k\text{-veces}} \longrightarrow \Lambda^k V$$

$$(v_1, \dots, v_k) \longmapsto v_1 \wedge \dots \wedge v_k$$

multiplicamos usando que $\wedge$ es alternante

$(\Lambda^k V, \wedge)$

espacio vec. — k-lineal alternante

① $\forall \sigma \in S_k$ $\quad v_1, \dots, v_k$

$$v_{\sigma(1)} \wedge \dots \wedge v_{\sigma(k)} = \operatorname{sgn}(\sigma)\, v_1 \wedge \dots \wedge v_k$$

② $\wedge$ es multilineal.

Lema: Sea $V = \langle e_1, \dots, e_n \rangle$ y sean $\vec{v}_1, \dots, \vec{v}_k \in V$

$$v_1 \wedge \dots \wedge v_k = \sum_{\substack{I \subseteq [n] \\ |I| = k}} \det(M_I)\, e_{i_1} \wedge \dots \wedge e_{i_k}$$

$I = \{i_1, \dots, i_k\}$ $\quad i_1 < i_2 < \dots < i_k$

$$M = \begin{array}{c} e_1 \\ \vdots \\ e_n \end{array} \underbrace{\begin{bmatrix} | & & | \\ v_1 & \dots & v_k \\ | & & | \end{bmatrix}}_{k}$$

M_I = submatriz cuadrada con filas indexadas por I

Ejercicio: Demostrar el lema.

Ejemplo: $V = \langle e_1, e_2, e_3 \rangle$

$\vec{u} = a e_1 + b e_2 + c e_3$

$\vec{v} = A e_1 + B e_2 + C e_3$

$$\vec{u} \wedge \vec{v} = (a e_1 + b e_2 + c e_3) \wedge (A e_1 + B e_2 + C e_3)$$

$$= \cancel{aA\, e_1 \wedge e_1} + aB\, e_1 \wedge e_2 + aC\, e_1 \wedge e_3 +$$
$$bA\, e_2 \wedge e_1 + \cancel{bB\, e_2 \wedge e_2} + bC\, e_2 \wedge e_3 +$$
$$cA\, e_3 \wedge e_1 + cB\, e_3 \wedge e_2 + \cancel{cC\, e_3 \wedge e_3}$$

$$= (aB - bA)\, e_1 \wedge e_2 + (aC - cA)\, e_1 \wedge e_3 + (bC - cB)\, e_2 \wedge e_3$$

$$M = \underbrace{\begin{bmatrix} a & A \\ b & B \\ c & C \end{bmatrix}}_{k} \Big\}\, n \qquad \binom{n}{k}$$

$$\begin{vmatrix} a & A \\ b & B \end{vmatrix} e_1 \wedge e_2 + \begin{vmatrix} a & A \\ c & C \end{vmatrix} e_1 \wedge e_3 + \begin{vmatrix} b & B \\ c & C \end{vmatrix} e_2 \wedge e_3$$

Ejercicio: Demuestre:

1. La prop universal del prod exterior lo determina de manera única.
2. Sobre la construcción demuestre
   a) $\dim(\Lambda^k V) = \binom{n}{k}$
   b) $(\Lambda^k V, \wedge)$ satisface la prop universal

Si $T : V \to W$ es lineal, definimos para $k \in \mathbb{N}$

$$\Lambda^k(T) : \Lambda^k V \longrightarrow \Lambda^k W \quad \text{así:}$$

$$(\vec{v}_1, \dots, \vec{v}_k) \longmapsto T(v_1) \wedge \dots \wedge T(v_k) := \wedge_W (T(v_1), \dots, T(v_k))$$

$$\begin{tikzcd}
\overbrace{V \times \dots \times V}^{k\text{-veces}} \arrow[r, "k\text{-lineal, alternante}"] \arrow[d, "\wedge_V"'] & \Lambda^k W \\
\Lambda^k V \arrow[ur, dashed, "\exists!\, \varphi =: \Lambda^k(T)"', "\text{lineal}"] &
\end{tikzcd}$$

$$\Lambda^k(T)(v_1 \wedge \dots \wedge v_k) = T(v_1) \wedge \dots \wedge T(v_k)$$

Def: Si (V, ρ_V) es una representación, definimos para $k \in \mathbb{N}$

$$(\Lambda^k V, \rho_{\Lambda^k(V)}), \quad \rho_{\Lambda^k(V)}(g) = \Lambda^k(\rho_V(g))$$

$V = \langle e_1, \dots, e_n \rangle$

$$V \xrightarrow{T} V \rightsquigarrow \begin{array}{c} \Lambda^n(V) \\ \parallel \\ \langle e_1 \wedge \dots \wedge e_n \rangle \end{array} \xrightarrow{\Lambda^n T} \begin{array}{c} \Lambda^n(V) \\ \parallel \\ \langle e_1 \wedge \dots \wedge e_n \rangle \end{array}$$

$[T]_B = A$

$$\Lambda^n(T)(e_1 \wedge \dots \wedge e_n) = T(e_1) \wedge \dots \wedge T(e_n) = \det(A)\, e_1 \wedge \dots \wedge e_n$$

Fórmula

$$\underbrace{\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix}}_{n} \Big\} n$$

$\Lambda^k T$

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[dr, "H"'] \arrow[dr, phantom, shift left=3, "\#"] & V \arrow[d, "F"] \\ & V \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} \Lambda^n V \arrow[r, "\Lambda^n T"] \arrow[dr, "\Lambda^n H"'] \arrow[dr, phantom, shift left=3, "\#"] & \Lambda^n V \arrow[d, "\Lambda^n F"] \\ & \Lambda^n V \end{tikzcd}$$

(Λ es functor)

$$\det([H]_B) = \det([F]) \cdot \det([T])$$

Cómo verificamos que es functor?

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[dr, "H"'] \arrow[dr, phantom, shift left=3, "\# \checkmark"] & W \arrow[d, "L"] \\ & S \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} \Lambda^k V \arrow[r, "\Lambda^k T"] \arrow[dr, "\Lambda^k H"'] \arrow[dr, phantom, shift left=3, "\overset{?}{\#}"] & \Lambda^k W \arrow[d, "\Lambda^k L"] \\ & \Lambda^k S \end{tikzcd}$$

$$\Lambda^k H(v_1 \wedge \dots \wedge v_k) = H(v_1) \wedge \dots \wedge H(v_k)$$

$$= L(T(v_1)) \wedge \dots \wedge L(T(v_k)) =$$

$$= \Lambda^k(L)\left(T(v_1) \wedge \dots \wedge T(v_k)\right) =$$

$$= \Lambda^k(L)\left(\Lambda^k T(v_1 \wedge \dots \wedge v_k)\right)$$

Ejercicios: Demuestre que, como representaciones

$$\Lambda^k(V^*) \cong \Lambda^k V^*$$

$$\Lambda^k(U \oplus W) = \bigoplus_{\substack{a,b \\ a+b=k}} \left[\Lambda^a U \oplus \Lambda^b W\right]$$

Ejercicio: Demuestre que, cno reps

$$V \otimes V \cong \mathrm{Sym}^2(V) \oplus \Lambda^2 V$$