Hoy: (Ejemplos de Caminatas aleatorias en grupos)

1) Sea $p \in \mathbb{N}$ p impar $\quad G = \mathbb{Z}/p\mathbb{Z}$

$$Q(g) = \begin{cases} \frac{1}{2}, & g = 1 \\ \frac{1}{2}, & g = -1 = p-1 \\ 0, & \text{d.l.c.} \end{cases} \qquad Q : G \longrightarrow \mathbb{C}$$

Definimos el siguiente proceso:

(1) $g_1, g_2, \dots,$ i.i.d. con dist Q

(2) $h_k := g_k + \dots + g_2 + g_1$

Pregunta: (1) Dist de h_k? $\quad \underbrace{Q * Q * \cdots * Q}_{H^{(k)}}(\alpha) = \mathbb{P}\{h_k = \alpha\}$

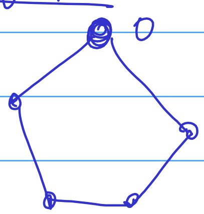
(2) Qué sucede si $k \to \infty$. En muchos casos $H^{(k)} \longrightarrow U$. Más precisamente

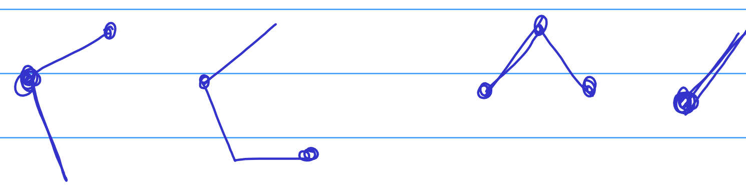
$$\| H^{(k)} - U \|_{TV} = \max_{A \subseteq G} \left| H^{(k)}(A) - U(A) \right| \xrightarrow[k \to \infty]{} 0$$

Qué tan rápido ocurre esto?

Ejemplo: $p = 5$, a mano calcular $H^{(3)}$

0





2^k # exp de términos

$$\widehat{Q_1 * Q_2}(\rho) = \hat{Q}_1(\rho) \cdot \hat{Q}_2(\rho)$$

prod de matrices

Solución: (1) $\overbrace{Q * Q * Q * \cdots * Q}^{k\text{-veces}}\,(\lambda)$ fácil

Idea: Aplico Fourier, calculo el producto, trasformada inversa de Fourier $= H^{(\lambda)}(\lambda)$.

Si ρ es irrep de G y $[\rho_V(g)]_B$ unitarias

$$\left[\hat{f}(\rho) = \sum_{g\in G} f(g)\,[\rho_V(g)]_B\right]$$

Recuerde

Como $\mathbb{Z}/p\mathbb{Z}$ es cíclico sus irreps son

$$\rho_j : \mathbb{Z}/p\mathbb{Z} \longrightarrow \mathbb{C}^*$$

$$1 \longmapsto \left[e^{i\frac{2\pi}{p}j}\right]$$

para $j=0,\dots,p-1$

unitarias

en rep ρ_j

$$\hat{Q}(\rho_j) = \sum_{g\in G} Q(g)\,\rho_j(g) = \tfrac{1}{2}\rho_j(1) + \tfrac{1}{2}\rho_j(-1)$$

$$= \tfrac{1}{2}e^{i\frac{2\pi}{p}j} + \tfrac{1}{2}e^{-i\frac{2\pi}{p}j} = \left[\cos\left(\tfrac{2\pi}{p}j\right)\right]$$

Ejercicio: (a) Demuestre que toda irrep es 1-dim'l $\Leftrightarrow$ G es abeliano.

(b) Encuentre los irreps de TODOS los gps abelianos finitos.

$$\widehat{H^{(k)}}(\rho_j) = \cos\left(\tfrac{2\pi}{p}j\right)^k$$

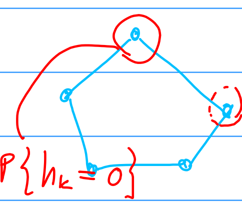
$$\left[H^{(k)}(g) = \frac{1}{|G|}\sum_{\rho \text{ irrep}} \dim(V_\rho)\,\mathrm{Tr}\left(\widehat{H^{(k)}}(\rho)\,\rho(g)^*\right)\right.$$

Formula de inv de Fourier

$$= \frac{1}{p}\sum_{j=0}^{p-1} 1\cdot \mathrm{Tr}\left(\cos\left(\tfrac{2\pi}{p}j\right)^k\,\overline{\rho_j(g)}\right)$$

$$\left.= \frac{1}{p}\sum_{j=0}^{p-1}\cos\left(\tfrac{2\pi}{p}j\right)^k\,\overline{\rho_j(g)}\right]$$

$$\boxed{H^{(k)}(0) = \frac{1}{p}\sum_{j=0}^{p-1}\cos\left(\tfrac{2\pi}{p}j\right)^k} \in \mathbb{Q}$$



$\mathbb{P}\{h_k = 0\}$

$$\left[\mathbb{P}\{ h_k = 1 \} = \frac{1}{p} \sum_{j=0}^{p-1} \cos\left(\frac{2\pi p}{j}\right)^k e^{-i \frac{2\pi}{p} j} \right] \in \mathbb{Q}.$$

1(2) $\left[\| H^{(k)} - U \|_{TV} \right] \overset{?}{\leq}$

Teorema: [Diaconis – Shahshahani upper bound]

Para cualquier distr de pes $Q : G \longrightarrow \mathbb{C}$ tenemos

$$\left[\| Q - U \|_{TV}^2 \leq \frac{1}{4} \sum_{\substack{\rho \neq \text{trivial} \\ \rho \text{ irrep}}} \dim(V_\rho) \, \underbrace{Tr\left(\hat{Q}(\rho) \hat{Q}(\rho)^* \right)}_{\| \hat{Q}(\rho) \|_{Fro}^2} \right]$$

Muy muy útil.

(b) $$\| Q - U \|_{TV}^2 \geq \frac{1}{|G|} \frac{1}{4} \sum_{\rho \neq \text{trivial}} \dim(V_\rho) \, Tr\left(\hat{Q}(\rho) \hat{Q}(\rho)^* \right)$$

Ejemplo:

$$\| H^{(k)} - U \|_{TV}^2 \leq \frac{1}{4} \sum_{j=1}^{p-1} 1 \, Tr\left(\hat{H}^{(k)}(\rho_j) \hat{H}^{(k)}(\rho_j)^* \right)$$

$$\hat{H}^{(k)}(\rho_j) = \cos\left(\frac{2\pi}{p} j\right)^k$$

$$= \underbrace{\frac{1}{4} \sum_{j=1}^{p-1} \cos\left(\frac{2\pi}{p} j\right)^{2k}}_{} \overset{\text{calculo}}{\leq} \frac{e^{-\frac{\pi^2}{p^2} k}}{2\left(1 - e^{-\frac{3\pi^2 k}{p^2}}\right)} \leq e^{-\frac{\pi^2}{p^2} k}$$

Idea: $\cos(x) \leq e^{-\frac{x^2}{2}}$, $x \in [0, \frac{\pi}{2}]$

si $k \geq p^2$

cutoff

"Despues de p^2 pasos la distancia se colapsa, pues decrece exponencialmente"

Dem: $$\| Q - U \|_{TV}^2 = \left(\frac{1}{2} \sum_{g \in G} | Q(g) - U(g) | \right)^2 = \frac{1}{4} \left(\sum_{g \in G} | Q(g) - U(g) | \right)^2$$

$\langle \vec{1}, (Q-U)_g \rangle^2$

$$\overset{\text{Cauchy Schwarz}}{\leq} \|1\|^2 \|Q-u\|_2^2 = |G|\,\|Q-u\|_2^2 \overset{\text{Identidad de Plancherel}}{=} \cancel{|G|}\left[\frac{1}{\cancel{|G|}} \sum_{\rho \text{ ireps}} \dim(V_\rho)\, \mathrm{Tr}\left((\hat{Q}-\hat{u}(\rho))(\hat{Q}-\hat{u}(\rho))^*\right)\right]$$

$$\text{Si } \rho = \text{triv} \quad (\hat{Q}-\hat{u})(\text{triv}) = \sum_{g\in G} Q(g)\cdot 1 - \underline{\sum_{g\in G} u(g)\cdot 1}$$

$$= 0$$

$$\text{Si } \rho_V \neq \text{triv} \quad \boxed{\hat{u}(\rho_V)} \overset{?}{=} \sum_{g\in G} \frac{1}{|G|} \rho_V(g)$$

$$= \left[\frac{1}{|G|} \sum_{g\in G} \underline{\rho_V}(g)\right] = \left(\pi_{\text{triv}} : V \longrightarrow V\right)$$

$$0$$

$$\left[\frac{1}{|G|} \sum_{g\in G} \rho_V(g) = \lambda\, \mathrm{Id}_V\right]$$

$$\frac{1}{|G|} \underset{\shortparallel}{\chi_V(g)} = \lambda \dim(V)$$

$$\langle \chi_{\text{triv}}, \chi_V \rangle = 0 \quad \Rightarrow \boxed{\lambda = 0}$$

$$= \sum_{\substack{\rho \text{ ireps} \\ \rho \neq \text{triv}}} \dim(V_\rho)\, \mathrm{Tr}\left(\hat{Q}(\rho)\, \hat{Q}(\rho)^*\right)$$