Hoy: Unicidad de descomposición $\rightsquigarrow$ Métodos efectivos de descomposición.

Continuando dem de Clase anterior...

$V \overset{\varphi}{\underset{\sim}{\leftarrow}} V_1^{\oplus a_1} \oplus \cdots \oplus V_k^{\oplus a_k}$ $\quad V_i$ irreps distintas

¿Qué de esta desc. es único?

(1) Los tipos de isom de las irreducibles que aparecen

(2) La multiplicidad a_i de cada V_i

[(3) Las componentes isotípicas $\varphi(V_i^{\oplus a_i})] \subseteq V$

Clase anterior (1) ✓ así que sabemos

$V \overset{\varphi}{\cong} [V_1^{\oplus a_1} \oplus \cdots \oplus V_k^{\oplus a_k} \overset{\psi}{\cong} V_1^{\oplus b_1} \oplus V_2^{\oplus b_2} \oplus \cdots \oplus V_k^{\oplus b_k}]$

Aver: $a_i = b_i \; \forall i$.

Mostraremos una descripción invariante bajo isomorfismo de reps del número a_i

Af 2: $a_i \overset{*}{=} \dim_{\mathbb{C}}(\mathrm{Hom}_G(V_i, V)) \Rightarrow a_i = b_i$

$V \xrightarrow[\text{isom}]{\psi} V'$

$\mathrm{Hom}_G(V_i, V) \xrightarrow{\cong} \mathrm{Hom}_G(V_i, V')$

$\varphi \longmapsto \psi \circ \varphi$

$\boxed{\text{Ejercicio}}$ Demuestre que, como espacios vectoriales hay isomorfismos:

$\mathrm{Hom}_G(A, B \oplus B') = \mathrm{Hom}_G(A, B) \oplus \mathrm{Hom}_G(A, B')$

$\mathrm{Hom}_G(A \oplus A', B) = \mathrm{Hom}_G(A, B) \oplus \mathrm{Hom}_G(A', B)$

Dem de (*): $\mathrm{Hom}_G(V_1, V)$ — irreducible

Ej

$\mathrm{Hom}_G(V_1, V_1^{\oplus a_1} \oplus V_2^{\oplus a_2} \oplus \cdots \oplus V_k^{\oplus a_k}) = \bigoplus_{i=1}^{k} \mathrm{Hom}_G(V_1, V_i)^{\oplus a_i}$

$$\text{Lema de Schur} \to \; = \; \underbrace{\mathrm{Hom}_q(V_1, V_1)}_{\text{e.v. 1-dimal}}^{\oplus a_i} \cong \mathbb{C}^{a_i} \Rightarrow a_i = \dim_{\mathbb{C}}\left(\mathrm{Hom}_q(V_i, V)\right)$$

(3) Hay un morfismo de reps:

$$\left[V_i \otimes \mathrm{Hom}_q(V_i, V) \xrightarrow{ev} V \right]$$

La componente isotípica $\varphi(V_i^{\oplus a_i}) = \mathrm{im}(ev)$. y ev es morfismo de reps con su imagen

$$V_i \otimes \mathrm{Hom}_q(V_i, V) \xrightarrow[\cong]{ev} V_i^{\oplus a_i}$$

Buena porque $V \xrightarrow{\psi} V'$

$$\begin{array}{ccc} V_i \otimes \mathrm{Hom}_q(V_i, V) & \longrightarrow & V \\ \downarrow{\scriptstyle id \otimes \psi} & \# & \downarrow{\scriptstyle \psi} \\ V_i \otimes \mathrm{Hom}_q(V_i, V') & \longrightarrow & V' \end{array}$$

Dem:

$$\begin{array}{ccc} V_i \times \mathrm{Hom}_q(V_i, V) & \xrightarrow[\text{bilineal}]{b_{ev}} & V \\ (\vec{w}, f) & \longmapsto & f(\vec{w}) \\ \downarrow{\scriptstyle \otimes} & \# \nearrow{\scriptstyle ev \text{ lineal}} & \\ V_i \otimes \underbrace{\mathrm{Hom}_q(V_i, V)}_{f^{\psi}} & & \end{array}$$

$$\begin{array}{ccc} V_i & \xrightarrow{f} & V \\ {\scriptstyle \rho_{V_i}(g)}\downarrow & \# & \downarrow{\scriptstyle \rho_V(g)} \\ V_i & \xrightarrow{f} & V \end{array}$$

(V_i, ρ_{V_i}) $\qquad \mathrm{Hom}_q(V_i, V) \subseteq \mathrm{Hom}(V_i, V)$

$(\mathrm{Hom}_q(V_i, V), \underline{\mathrm{triv}})$ $\qquad (V, \rho_V)$

Af: ev es $\underline{\text{morfismo de reps}}$

$$\underline{ev}\left(\rho_{V_i \otimes \mathrm{Hom}_q(V_i, V)}(g)\,(\vec{w} \otimes f)\right) = ev\left(\rho_{V_i}(g)(\vec{w}) \otimes f\right)$$

$$= f\left(\rho_{V_i}(g)\,(\vec{w})\right) = \rho_V(g)\left(f(\vec{w})\right) = \rho_V(g)\left(ev(w \otimes f)\right)$$

$$ev \in \mathrm{Hom}_q\left(V_i \otimes \mathrm{Hom}_q(V_i, V),\ V\right)$$

$$V \overset{\varphi}{\cong} V_1^{\oplus a_1} \oplus \cdots \oplus V_k^{\oplus a_k}$$

$a_1 \dim(V_1)$

$$V_1 \otimes \underbrace{\mathrm{Hom}_G(V_1, V)}_{} \xrightarrow{ev} V_1$$

$$\boxed{\varphi_0^{-1}\, \mathrm{im}(ev) \doteq V_1^{\oplus a_1} \oplus \vec{0}} \subset V$$

como ambos espacios tienen la misma dim ev es un isomorfismo!

$$V_1 \otimes \mathrm{Hom}_G(V_1, V) \underset{\cong}{\xrightarrow{ev}} V_1^{\oplus a_1}$$

de reps!

$$V_1 \xrightarrow[\text{de reps}]{\substack{i_j \\ \text{morfismo}}} \overbrace{V_1 \oplus \cdots \oplus V_1}^{a_1} \xrightarrow{\varphi \circ i} V$$

$1 \leq j \leq a_1$

$$i_j \in \underline{\mathrm{Hom}_G(V_1, V)}$$

$$\mathrm{im}(ev) \supseteq V_1^{a_1} \oplus \vec{0}$$

$$V_1 \otimes \mathrm{Hom}_G(V_1, V)$$

$$(\vec{w}, i_j) \longmapsto^{ev}$$

Obs:

$$\begin{array}{ccc} V_1^{\oplus a_1} \oplus \left(\bigoplus_{k \neq 1} V_k^{\oplus a_k}\right) & \xrightarrow{\pi_2} & \bigoplus_{k\neq 1} V_k^{\oplus a_k} \\ \uparrow{\scriptstyle ev} & \nearrow & \\ V_1 \otimes \mathrm{Hom}_G(V_1, V) & 0 & \end{array}$$

$$\boxed{(\mathbb{C}^2, \mathrm{triv.}) \underset{\text{como ev.}}{\cong} (\mathbb{C}^2, \mathbb{Z}/2\mathbb{Z})}$$

$g \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\quad$ $1 \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

NO como reps de $\mathbb{Z}/2\mathbb{Z}$

$$\rho(\mathcal{I}) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Ejemplo: $\mathbb{Z}/2\mathbb{Z} = \{e, \mathcal{I}\} \curvearrowright \langle e_1, e_2 \rangle = V$

$$\left[V_1 = \langle u \rangle, \quad \rho_{V_1}(\mathcal{I}) = -1 \right]$$

¿Cuántas copias de signo (V_1) hay en V?

¿Dónde están?

$$V_1 \otimes \mathrm{Hom}_G(V_1, V) \xrightarrow{ev} V$$

$$\mathrm{Hom}_G(V_1, V) \subseteq \mathrm{Hom}(V_1, V)$$

$$\begin{array}{c} u \\ \begin{array}{c} e_1 \\ e_2 \end{array} \begin{bmatrix} A \\ B \end{bmatrix} \end{array}$$

$$\begin{tikzcd} V_1 \arrow[r, "T"] \arrow[d, "\rho_{V_1}(\mathcal{I})"'] \arrow[dr, phantom, "\#"] & V \arrow[d, "\rho_V(\mathcal{I})"] \\ V_1 \arrow[r, "T"] & V \end{tikzcd}$$

$$\left[T \circ \rho_{V_1}(\mathcal{I}) = \rho_V(\mathcal{I}) \circ T \right] \quad \text{Lineal en } T \text{ !!}$$

$$\left[\begin{bmatrix} A \\ B \end{bmatrix} [-1] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} \right]$$

$$f(u) = e_1 - e_2$$

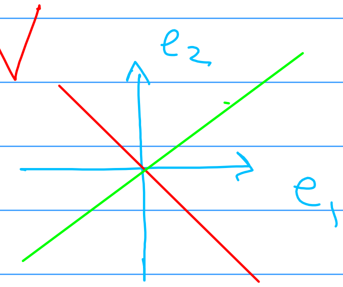
$$\begin{bmatrix} -A \\ -B \end{bmatrix} = \begin{bmatrix} B \\ A \end{bmatrix} \Rightarrow B = -A$$

$$\mathrm{Hom}_G(V_1, V) = \left\{ \begin{bmatrix} A \\ B \end{bmatrix} : B = -A \right\} = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$$

f

$$(w, \varphi) \longmapsto \varphi(w)$$

$$\underbrace{V_1} \otimes \underbrace{\mathrm{Hom}_g(V_1, V_1)} \xrightarrow{ev} V$$



$$\langle u \otimes f \rangle$$

$$ev(u \otimes f) = f(u)$$

$$im(ev) = \lambda f(u) = \boxed{\lambda(e_1 - e_2)} \checkmark$$

$$\mathrm{Hom}_g(\mathrm{triv}, V)$$

$$\mathrm{triv} = \langle t \rangle$$

$$\mathrm{Hom}(\mathrm{triv}, V) = \left\{ \begin{array}{c|c} & t \\ e_1 & \multirow{2}{*}{$\begin{bmatrix} A \\ B \end{bmatrix}$} \\ e_2 & \end{array} : A, B \in \mathbb{C} \right\}$$

$$\begin{tikzcd} \mathrm{tr} \arrow[r, "T"] \arrow[d, "\rho_{tr}(J)"'] \arrow[dr, phantom, "\#"] & V \arrow[d, "\rho_V(J)"] \\ \mathrm{tr} \arrow[r, "T"] & V \end{tikzcd}$$

$$\begin{bmatrix} A \\ B \end{bmatrix} [1] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix}$$

$$\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} B \\ A \end{bmatrix} \Rightarrow A = B$$

$$\begin{array}{cc} & t \\ e_1 & \multirow{2}{*}{$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$} \\ e_2 & \end{array}$$

$$\mathrm{Hom}_g(\mathrm{triv}, V) = \left\{ \begin{bmatrix} A \\ A \end{bmatrix} : A \in \mathbb{C} \right\}$$

$$\mathrm{triv} \otimes \mathrm{Hom}_g(\mathrm{triv}, V) \xrightarrow{ev} V \qquad im(ev) = \mathrm{span}(e_1 + e_2)$$