Hoy: Grassmannianas

$$\left\{\begin{matrix}\text{subespacios } k\text{-dim}\\ \text{de } V\end{matrix}\right\} \xrightarrow{\overset{\text{Plücker embedding}}{P}} \mathbb{P}(\Lambda^k V)$$

$$W = \langle v_1, \dots, v_k\rangle \longrightarrow [v_1 \wedge \dots \wedge v_k]$$

$$Gr(k, V) := im(P) = \{[\omega] : \underset{\text{tot desc.}}{\omega = v_1 \wedge \dots \wedge v_k}\}$$

Si $[\omega] \in \mathbb{P}(\Lambda^k V)$, $L_{[\omega]} := \{v \in V : \underline{\omega \wedge v = 0}\}$

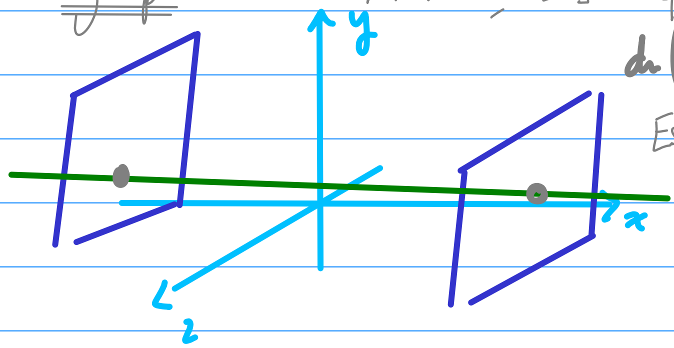
Lema: $\dim(L_{[\omega]}) \geq \boxed{k} \iff [\omega] \in Gr(k, V)$

y en ese caso $\dim(L_{[\omega]}) = k$.

$\omega \wedge v = 0 \iff \exists \alpha : \omega = \alpha \wedge v$

Obs: El lema nos da ecuaciones para $Gr(k,V)$.

Ejemplo: $\omega \in \Lambda^2 \mathbb{C}^4$, $[\omega] \in Gr(2, \mathbb{C}^4) = Gr(1, \mathbb{P}^3)$



$\dim(\mathbb{P}(\Lambda^2\mathbb{C}^4)) = 5$

Esperamos que $I(Gr(2,\mathbb{C}^4)) = (Q)$.

$$\Lambda^2 \mathbb{C}^4 = \langle e_1\wedge e_2, e_1\wedge e_3, e_1\wedge e_4, e_2\wedge e_3, e_2\wedge e_4, e_3\wedge e_4\rangle$$

$$[\omega = p_{12}(e_1\wedge e_2) + p_{13}(e_1\wedge e_3) + \dots + p_{34}(e_3\wedge e_4)]$$

$$V = \langle e_1, e_2, e_3, e_4\rangle \qquad V \xrightarrow[\text{lineal}]{\varphi_\omega} \Lambda^3 V = \langle e_2\wedge e_3\wedge e_4,\ e_1\wedge e_3\wedge e_4,\ e_1\wedge e_2\wedge e_4,\ e_1\wedge e_2\wedge e_3\rangle$$

$$v \longmapsto v \wedge \omega$$

$$\begin{array}{c|cccc} & e_1 & e_2 & e_3 & e_4 \\ \hline e_{234} & 0 & p_{34} & -p_{24} & p_{23} \\ e_{134} & p_{34} & 0 & -p_{14} & p_{13} \\ e_{124} & p_{24} & -p_{14} & 0 & p_{12} \\ e_{123} & p_{23} & -p_{13} & p_{12} & 0 \end{array}$$

$e_4 \wedge \omega = p_{12}\, e_4 \wedge e_1 \wedge e_2 + p_{13}\, e_4 \wedge e_1 \wedge e_3 + p_{12}\, \underbrace{e_4 \wedge e_2 \wedge e_3}$

$- e_2 \wedge e_4 \wedge e_3$

$e_2 \wedge e_3 \wedge e_4$

$$e_1 \wedge \omega = p_{23}\, e_1 \wedge e_2 \wedge e_3 + p_{24}\, e_1 \wedge e_2 \wedge e_4 + p_{34}\, e_1 \wedge e_3 \wedge e_4$$

$$\dim(\ker(\varphi_\omega)) \geq 2 \iff \dim(\mathrm{Im}(\varphi_\omega)) \leq 2$$

$$3 \times 3 \text{ mins } (\varphi_\omega) = 0$$

16 ecs. cúbicas.

$V\ (n-k+1 \text{ mins } (\varphi_\omega))$ ✓

Conclusión: (1) $V\left(3 \times 3 \text{ mins } (\varphi_\omega)\right) = Gr(1, \mathbb{P}^3)$

así que $Gr(1, \mathbb{P}^3)$ es una v.a.

(2) Parece que estas ecuaciones no dan $I(Gr(k, \mathbb{P}^n))$.

Ejercicio:

(1) Fije iso entre $\Lambda^n V \xrightarrow{\sim} \mathbb{C}$

(a) $\Lambda^k V \rightsquigarrow [\Lambda^{n-k} V^*]$, $\omega \longmapsto [\omega^*]$ ← natural "up to scalars"

$\Lambda^{n-k} V \longrightarrow \Lambda^n V \xrightarrow{\sim} \mathbb{C}$

$\alpha \longmapsto \alpha \wedge \underline{\underline{\omega}}$

$\Lambda^k V \longrightarrow \mathrm{Hom}(\Lambda^{n-k} V, \mathbb{C})$

$(\Lambda^{n-k} V)^* = \Lambda^{n-k} V^*$

(b) $\varphi(\omega): V \longrightarrow \Lambda^{k+1} V$

$v \longmapsto v \wedge \omega$

$\Psi(\omega^*): V^* \longrightarrow \Lambda^{n-k+1} V^*$

$v^* \longmapsto v^* \wedge \omega^*$

* Af: ω es descomponible $\iff \mathrm{Ker}(\varphi(\omega)) \boxed{=} \mathrm{Ker}(\Psi(\omega))^\perp$

(c) Equiv. $\forall \alpha \in \Lambda^{k+1} V^*$, $\beta \in \Lambda^{n-k+1} V$

$$\Xi_{\alpha,\beta}(\omega) = \langle \varphi^t(\underset{\smile}{\omega})(\alpha), \psi^t(\underset{\smile}{\omega})(\beta) \rangle = 0$$

← Relaciones de Plücker.

Verifique que es una ecuación cuadrática en p_{ij}'s.

(d) Encuentre el polinomio que define $Gr(1, \mathbb{P}^3) \subseteq \mathbb{P}(\Lambda^2 \mathbb{C}^4)$.

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[dr, "\ell \circ \varphi"'] & B \arrow[d, "\ell"] \\ & \mathbb{C} \end{tikzcd} \rightsquigarrow \begin{tikzcd} B^* \arrow[r, "\varphi^t"] & A^* \end{tikzcd}$$

$\varphi^t(\ell) := \ell \circ \varphi$

$\mathrm{im}(\varphi)^\perp = \ker(\varphi^t)$

$\{\ell \in B^* : \ell(\mathrm{im}(\varphi)) = 0\}$

(2)

Cómo es localmente $Gr(k,V) \subseteq \underline{\mathbb{P}(\Lambda^k V)}$?

Sabemos que $\mathbb{P}^n = \bigcup_{j=0}^{n} U_j$ donde $U_j = \{[x] : x_j \neq 0\}$

$$[\underset{\neq 0}{x_0} : x_1 : \dots : x_n] \in \underset{\substack{\wr\wr \\ \mathbb{C}^n}}{U_0}$$

$$\left[1 : \frac{x_1}{x_0} : \dots : \frac{x_n}{x_0}\right]$$

$$\Big\downarrow$$

$$\left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right) \cong \boxed{\mathbb{C}^n}$$

En coordenadas el mapa de Plück funciona así:

$$\left\{ \begin{array}{c} \text{Subespacios de dim} \\ k \quad W \subseteq V \end{array} \right\} \longrightarrow \mathbb{P}(\Lambda^k V)$$

$W = \langle v_1, \dots, v_k \rangle$ $\qquad$ $\langle e_{i_1} \wedge \dots \wedge e_{i_k} : 1 \le i_1 < i_2 < \dots < i_k \le n \rangle$

$$\underset{A}{k\left\{\right.} \begin{array}{cccc} e_1 & e_2 & \dots & e_n \\ \hline \multicolumn{4}{c}{v_1} \\ \multicolumn{4}{c}{v_2} \\ \multicolumn{4}{c}{\vdots} \\ \multicolumn{4}{c}{v_k} \end{array} \longmapsto \left[\sum_{\substack{I \subseteq [n] \\ |I| = k}} p_I \, e_I \right]$$

$\underbrace{\qquad\qquad}_{n}$

algún menor $k \times k$ es distinto de cero.

columnas del conjunto I

$$p_I(A) = \det(\overline{A_I})$$

Si $\quad p_{1\dots k} \neq 0 \quad$ entonces

$A = [\boxed{D} \quad R]$ tiene el mismo sp de filas que

$$D^{-1}A = [I \quad B] = \left[\underbrace{\begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ 0 & 0 & \dots & 1 \end{bmatrix} \boxed{\begin{bmatrix} a_{11} & \dots & a_{1,n-k} \\ \vdots & & \\ a_{k1} & \dots & a_{k,n-k} \end{bmatrix}}}_{} \right] \quad B \in \mathbb{C}^{k(n-k)}$$

Obs: Las coordenadas de Plück de un espacio de esta forma contienen a las entradas de B

$P_{12..k} = 1$

$P_{k+i,2,..,k} = a_{11}$

$P_{k+j,1..i-1,i+1..k} = a_{ij}$

$[1 : B : F(B)]$

$$\begin{tikzcd} Gr(k,V) \cap U \arrow[d, shift right=2, "\pi_B"'] \\ \mathbb{C}^{k(n-k)} \arrow[u, shift right=2, "\cong"'] \end{tikzcd} \quad !! \qquad = \left\{ (B, F(B)) \in \mathbb{C}^{\binom{n}{k}-1} \right\}$$

$\underbrace{P_{12..k} \neq 0}$

$Gr(k, \mathbb{C}^n) \cong$ Union de $\binom{n}{k}$ copias de $\mathbb{C}^{k(n-k)}$.

Ejemplo. $Gr(1, \mathbb{P}^3) = Gr(2, \mathbb{C}^4)$

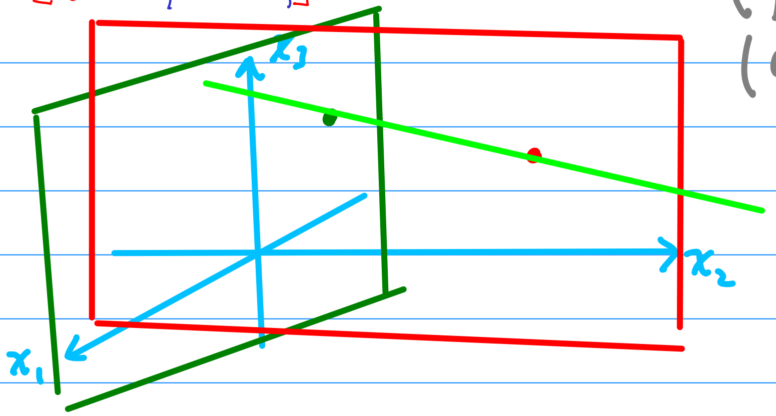
$$\left[\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{cccc} e_1 & e_2 & e_3 & e_4 \\ \hline 1 & 0 & a & b \\ 0 & 1 & c & d \end{array}$$

Si $x_4 \neq 0$

$\left(\frac{1}{b}, 0, \frac{a}{b}\right)$

$\left(0, \frac{1}{d}, \frac{c}{d}\right)$



(3) Cómo es categóricamente? Hay alguna propiedad universal que describa morfismos

$$\begin{cases} Z \xrightarrow{\varphi} Gr(k,V) & \text{ó} \\ Gr(k,V) \xrightarrow{\psi} Z & ? \end{cases}$$

(3.1) La familia Universal de k-planos arriba de $Gr(k, \mathbb{C}^n)$ es

Haz vect de rango k.

$$Gr(k, \mathbb{C}^n) \times \mathbb{C}^n \supseteq \begin{tikzcd} U \arrow[d, "\pi_1"] \\ Gr(k,\mathbb{C}^n) \end{tikzcd} = \{ (\underbrace{[w]}, v) : \underbrace{v \in L}_{w} \}$$

(1) $\mathcal{U}$ es una subvariedad de $G \times \mathbb{C}^n$

(2) $\mathcal{U} \xrightarrow{\pi_1} G$ es un haz vectorial de rango k.

Def: Una pareja (E, π) es un haz vectorial de rango r.

(1) E es variedad y $\pi : E \longrightarrow X$ es morfismo.

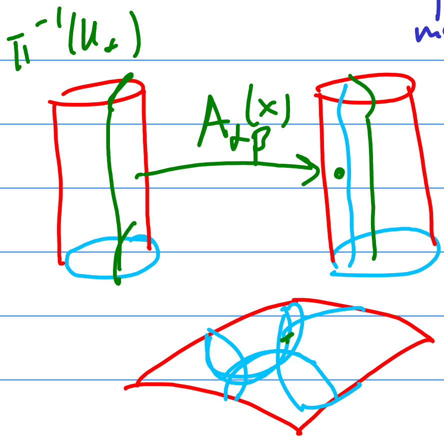
(2) Existe una cubierta por abiertos de X $\{U_\alpha\}_{\alpha \in I}$ e isomorfismos ψ_i:

$$\begin{tikzcd}
\pi^{-1}(U_\alpha) \arrow[r, "\psi_\alpha"] \arrow[dr, "\pi"'] & U_\alpha \times \mathbb{C}^{\boxed{r}} \arrow[d, "\pi_1"] \arrow[dl, phantom, "\#"] \\
 & U_\alpha
\end{tikzcd}$$

(3) $\psi_\beta \circ \psi_\alpha^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{C}^n \xrightarrow{\psi_\alpha^{-1}} \pi^{-1}(U_\alpha \cap U_\beta) \xrightarrow{\psi_\beta} (U_\beta \cap U_\alpha) \times \mathbb{C}^n$

$$\psi_\beta \circ \psi_\alpha^{-1}\left((\underline{x}, v) \right) = \left(\underline{x}, \boxed{A_{\alpha\beta}(x)} \vec{v} \right)$$

matriz.



Arriba de $U_{p_{12\ldots k} \neq 0}$

$\underline{U_{p_{1\ldots k} \neq 0} \times \mathbb{C}^k}$

$\mathbb{C}^{k(n-k)} \times \mathbb{C}^k$

$\cong$

$A = \begin{bmatrix} I & B \end{bmatrix}$

$$\left\{ \left([w], \underset{\cap}{w} \right) \right\} \cong \left\{ \left(B, (c_1, \ldots, c_k) \right) \right\}$$

sp de filas de A

$$w = c_1 f_1 + \dots + c_k f_k = (c_1, \dots, c_k], \tilde{c}_{k+1}, \dots, \tilde{c}_n)$$

Cómo cambian las coordenadas cuando hacemos un cambio de carta ?

Ejemplo:

$$\begin{bmatrix} 1 & 0 & A & B \\ 0 & 1 & C & D \end{bmatrix} \quad \cap \quad \left[\begin{array}{c|cc|c} E & 1 & 6 & G \\ F & 0 & 1 & H \end{array}\right]$$

$$\uparrow \qquad\qquad\qquad \downarrow$$

$$\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}, \begin{pmatrix} a \\ b \end{pmatrix} \right)$$

$$U \ni \left(\left[\begin{array}{c|cc|c} 1 & 0 & A & D \\ 0 & 1 & C & D \end{array}\right], \underline{(\underline{a}, \underline{b}, \underline{aA+bC}, aB+bD)} \right)$$

$$\frac{1}{-A}\begin{bmatrix} C & -A \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & A & B \\ 0 & 1 & C & D \end{bmatrix} = \begin{bmatrix} -\frac{C}{A} & 1 & 0 & \frac{CB-AD}{-A} \\ \frac{1}{A} & 0 & 1 & \frac{B}{A} \end{bmatrix}$$

$$(b, aA+bC)$$

$$\shortparallel$$

$$\boxed{\begin{pmatrix} 0 & 1 \\ A & C \end{pmatrix}} \cdot \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\uparrow$$

$$\left\{ \underline{\begin{pmatrix} A & D \\ C & D \end{pmatrix}}, \begin{pmatrix} a \\ b \end{pmatrix} \right\} \longmapsto \left\{ \underline{\begin{pmatrix} -\frac{C}{A} & \frac{CD-AD}{-A} \\ \frac{1}{A} & \frac{D}{A} \end{pmatrix}}, \begin{pmatrix} 0 & 1 \\ A & C \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \right\}$$

Teorema: Hay una correspondencia biyectiva entre

$$\left\{ \begin{array}{c} \text{morfismos} \\ Z \longrightarrow Gr(k, \mathbb{C}^n) \end{array} \right\} \longleftrightarrow \left[\left\{ \begin{array}{l} \text{Subhaces vectoriales} \\ \text{de rango } k \\ \text{de } \ Z \times \mathbb{C}^n \end{array} \right\} \right]$$

$$\varphi \longmapsto \varphi^*(\mathcal{U})$$