$\exists\, G$ y (V, ρ_G) es una rep de G.

De la clase anterior: $V = \langle e_1, \dots, e_n \rangle$

Si $X \subseteq \mathbb{C}^n$ admite simetrías $\Rightarrow$ ($\forall g \in G\ (\rho(g)(X) \subseteq X)$)

$\mathcal{I}(X) \subseteq \mathbb{C}[x_1, \dots, x_n]$ es una subrepresentación de $\mathrm{Sym}^\bullet(V)$

Ejemplo: Caso de matrices de rango 1.

Matrices $\in A^* \otimes B^*$

$$x_{ij} \longmapsto \sum_{i,j} x_{ij} (\alpha_i \otimes \beta_j) \qquad (x_{11}, \dots, x_{nn})$$

Queremos polinomios que se desvanezcan en los de rango 1.

$(A^* \otimes B^*)^*$ ← polinomios de grado 1 en mi espacio

$$\Vert$$

$$(A \otimes B) \ni \sum_{i,j} c_{ij}\, a_i \otimes b_j \qquad \begin{pmatrix} \alpha_i \text{ dual de } a_i \\ \beta_j \text{ dual de } b_j \end{pmatrix}$$

$$\left(\sum_{i,j} c_{ij}\, a_i \otimes b_j\right)\left(\sum_{s,t} x_{st}\, \alpha_s \otimes \beta_t\right) = \left[\sum_{i,j} c_{ij}\, x_{ij}\right]$$

Recuerda que:

$$a_i \otimes b_j\,(\alpha_s \otimes \beta_t) = \alpha_s(a_i) \cdot \beta_t(b_j) \in \mathbb{C}$$

Queremos ecuaciones de grado mayor; para ello

(1) Descomponemos $\mathrm{Sym}^k(A \otimes B)$ en irreducibles (no isomorfos)

$$\Vert$$

$$(V_1^{\oplus a_1}) \oplus \dots \oplus V_j^{\oplus a_j}$$

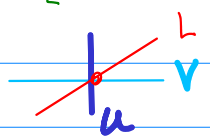
Como $\mathcal{I}(X)$ es una subrep de $\mathrm{Sym}^\bullet(V)$

(2) $\mathcal{I}(X)_k$ es la generado en grado k

$$\to \left[\mathcal{I}(X) \cap (V_j^{\oplus a_j})\right]$$

Ejemplo:

$\mathbb{R}^2 = U \oplus V$



$$L = L \cap \mathbb{R}^2 \neq (L \cap U) \oplus (L \cap V)$$

$$\begin{array}{ccc} V & \xrightarrow{T} & W \\ \rho_V(g)\downarrow & \# & \downarrow \rho_W(g) \\ V & \xrightarrow{T} & W \end{array}$$

Como reps de $G = GL(A) \times GL(B)$

Lema: $\mathrm{Sym}^2(A\otimes B) \cong [S^2(A) \otimes S^2(B)] \oplus [\Lambda^2 A \otimes \Lambda^2 B]$

irrep $\neq$ irrep

Hecho: $\forall A:$ $[A \otimes A \cong [\mathrm{Sym}^2(A)] \oplus [\Lambda^2 A]]$ como reps de $GL(A)$

Dem:

$$a\otimes b = \left[\frac{1}{2}[a\otimes b + b\otimes a]\right] + \frac{1}{2}[a\otimes b - b\otimes a]$$

simétrico

$\frac{1}{2}[b\otimes a - a \otimes b] = -\frac{1}{2}[a\otimes b - b\otimes a]$ antisimétrico

$g \in GL(A)$

$i: \mathrm{Sym}^2(A) \longrightarrow A\otimes A$

$\mathrm{Sym}^2(A) \subseteq A\otimes A$ es una subrep.

$T \in \mathrm{Sym}^2(A)$, $g \in GL(A)$

$$T = \sum_{(a,b)} c_{ab}\left(\frac{1}{2}(a\otimes b + b\otimes a)\right)$$

$$g(T) = \sum c_{ab}\left(\frac{1}{2}\, g(a)\otimes g(b) + g(b)\otimes g(a)\right) \checkmark$$

simétrico

$\mathrm{Sym}^2(A) =$ "Polinomios de grado 2 en variables $a_1, \dots, a_n$"

$\|?$

$$\mathrm{Sym}^2(A)' = \left\langle \frac{1}{2}(a\otimes b + b\otimes a)\right\rangle_{\{a,b\in A\}}$$

$$\begin{array}{ccc} \mathrm{Sym}^2(A) & \longrightarrow & \mathrm{Sym}^2(A)' \\ a_1 \cdot a_2 & \longmapsto & \frac{1}{2}(a_1\otimes a_2 + a_2 \otimes a_1) \end{array}$$

$$Sym^2(A\otimes B) \subseteq [(A\otimes B)\otimes(A\otimes B)]$$

$$\cong [(A\otimes A)\otimes(B\otimes B)]$$

$$= [Sym^2(A) \oplus \Lambda^2 A] \otimes [Sym^2(B) \oplus \Lambda^2(B)]$$

$$= [Sym^2(A)\otimes Sym^2(B)] \oplus [\Lambda^2 A \otimes \Lambda^2 B] \oplus \text{"otros dos términos"}$$

$Sym^2(A\otimes B)$.

$G = GL(A)\times GL(B)$ (g_A, g_B)

$\Lambda^2 A \ni$

$$\left[\tfrac{1}{2}(a_1\otimes a_2 \pm a_2\otimes a_1)\otimes \tfrac{1}{2}(b_1\otimes b_2 \pm b_2\otimes b_1)\right]$$

$A\otimes A \otimes B\otimes B$

morfismo de G-reps.

$A\otimes A\otimes B\otimes B$

$$\tfrac{1}{4}[a_1a_2b_1b_2 \pm a_1a_2b_2b_1 \pm a_2a_1b_1b_2 + a_2a_1b_2b_1]$$

$\downarrow$ isom

$A\otimes B\otimes A\otimes B$

$$\tfrac{1}{4}[(a_1b_1)a_2b_2 \pm a_1b_2a_2b_1 \pm a_2b_1a_1b_2 + a_2b_2a_1b_1]$$

Simétrico? sí!

$\leftarrow (g_A, g_B)$

$$\tfrac{1}{4}[a_2b_2\,a_1b_1 \pm a_2b_1\,a_1b_2 \pm a_1b_2\,a_2b_1 + a_1b_1\,a_2b_2]$$

$$\in Sym^2(A\otimes B) \subseteq (A\otimes B)\otimes(A\otimes B)$$

$$\left[\tfrac{1}{4}\left[(a_1\otimes b_1)\otimes(a_2\otimes b_2) - (a_1\otimes b_2)\otimes(a_2\otimes b_1) + (a_2\otimes b_2)\otimes(a_1\otimes b_1) - (a_2\otimes b_1)\otimes(a_1\otimes b_2)\right]\right]$$

$$\left[\tfrac{1}{2}\left[(a_1\otimes b_1)\cdot(a_2\otimes b_2) - (a_1\otimes b_2)\cdot(a_2\otimes b_1)\right]\right] \varphi$$

$$W \ni \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = (x_{11}\,\alpha_1\otimes\beta_1 + x_{12}\,\alpha_1\otimes\beta_2 + x_{21}\,\alpha_2\otimes\beta_1 + x_{22}\,\alpha_2\otimes\beta_2)$$

$$[\varphi(W) = [x_{11}x_{22} \pm x_{12}x_{21}]]$$

Hemos construido morfismos no nulos

$$\left[\Lambda^2(A)\otimes\Lambda^2(B)\right] \xrightarrow{\varphi_1} \mathrm{Sym}^2(A\otimes B)$$

$$\left[S^2(A)\otimes S^2(B)\right] \xrightarrow{\varphi_2} \mathrm{Sym}^2(A\otimes B)$$

(≠ between $\Lambda^2(A)$ and $S^2(A)$; 1–1)

Como son rep irreducibles todo morfismo no nulo es inyección porque $\ker(\varphi)$ es una subrepresentación.

Falta ver que $\mathrm{Sym}^2(A\otimes B) = \mathrm{im}(\varphi_1)\oplus\mathrm{im}(\varphi_2)$

(a) $\mathrm{im}(\varphi_1)\cap\mathrm{im}(\varphi_2) = \{0\}$ ✓

(b) $(\langle \mathrm{im}(\varphi_1), \mathrm{im}(\varphi_2)\rangle) = \mathrm{Sym}^2(A\otimes B)$

$[\dim(\langle A, B\rangle) = \dim(A) + \dim(B) - \dim(A\cap B)$

Cuál es la dimensión de: $\dim(A)=a$, $\dim(B)=b$

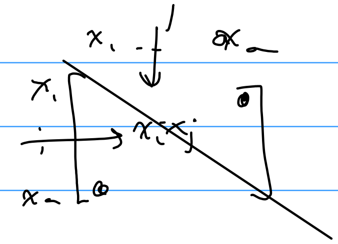
$\dim(\Lambda^2 A) = \binom{a}{2}$ $\qquad \binom{a}{2}\cdot\binom{b}{2} \neq \binom{a+1}{2}\cdot\binom{b+1}{2}$

$\dim(\Lambda^2 B) = \binom{b}{2}$

$\dim(\mathrm{Sym}^2(A)) = \binom{a+1}{2}$

$\dim(\mathrm{Sym}^2(B)) = \binom{b+1}{2}$

$\dim(\mathrm{Sym}^2(A\otimes B)) = \binom{ab+1}{2}$



$\frac{a^2-a}{2} + a = \frac{a^2+a}{2} = \binom{a+1}{2}$

$\frac{a(a-1)}{2}$

$\dim((S^2(A)\otimes S^2(B))) = \binom{a+1}{2}\cdot\binom{b+1}{2}$ $\qquad \binom{ab+1}{2}$

$\dim(\Lambda^2 A\otimes\Lambda^2 B) = \binom{a}{2}\cdot\binom{b}{2}$

verificar → (11)

$\dim(S^2(A)\otimes S^2(B)\oplus\Lambda^2 A\otimes\Lambda^2 B) = \binom{a+1}{2}\binom{b+1}{2} + \binom{a}{2}\binom{b}{2}$

Por qué son suficientes?

$A \xrightarrow{f} B^*$

$\Lambda^2 A \xrightarrow{\Lambda^2 f} \Lambda^2 B^*$

Sabemos que $[\Lambda^2 f \equiv 0] \iff \text{rango}(f) = 1$

$\Lambda^2 f \in Hom(\Lambda^2 A, \Lambda^2 B^*) = (\Lambda^2 A)^* \otimes \Lambda^2 B^* =$

$\Lambda^2 f \in [\Lambda^2 A^* \otimes \Lambda^2 B^*]$

decir: este elemento es cero.

Hecho: Si U es un ev cualquiera hay un emparejamiento así

$U^* \times U \longrightarrow \mathbb{C}$

$(f, u) \longmapsto f(u)$

este emparejamiento es NO DEGENERADO

$[\{u \in U : f(u) = 0 \ \forall f \in U^*\} = \{0\}]$

$u \in U \quad u = c_1 e_1 + \dots + c_n e_n$

$e_j^*(u) = c_j$

$(\Lambda^2 A^* \otimes \Lambda^2 B^*)^* = \boxed{\Lambda^2 A \otimes \Lambda^2 B}$ y por eso esas ecuaciones son suficientes.

$V(\Lambda^2 A \otimes \Lambda^2 B) = \{\text{matrices de rango } 1\}$

$\Lambda^2 A \otimes \Lambda^2 B \subseteq Sym^2(A \otimes B)$

Aplicando I a ambos lados:

$\sqrt{(\Lambda^2 A \otimes \Lambda^2 B)} = I(\{\text{matrices de rango } 1\})$

$\|$ ← Teorema: El ideal generado por los menores 2x2 es radical (Sturmfels "Algorithms in Invariant Theory")

$(\Lambda^2 A \otimes \Lambda^2 B)$

Como representación de $GL(A) \times GL(B)$?

Ejercicio: $\Lambda^2(A \otimes B) \overset{?}{=}$