El Teorema de Alexander-Hirschowitz responde

$$\dim\left(\sigma_k(\nu_d(\mathbb{P}^n))\right) = \min\left(k\dim(\nu_d(\mathbb{P}^n)) + (k-1),\ \binom{n+d}{d}-1\right)$$

salvo por puntos excepcionales que el teorema explica.

Concluimos que * casi siempre *

$$\dim\left(\sigma_k(\nu_d(\mathbb{P}^n))\right) = k\cdot n + k-1 = k(n+1)-1$$

$$\nu_d(\mathbb{P}^n) \subseteq \mathbb{P}(\mathrm{Sym}^d(V)) = \mathbb{P}^{\binom{n+d}{d}-1}$$

$$\sigma_1 \subsetneq \sigma_2 \subsetneq \sigma_3 \subsetneq \cdots \subsetneq \sigma_N = \mathbb{P}^{\binom{n+d}{d}-1}$$

Llenamos el espacio ambiente cuando

$$k(n+1)\cancel{-1} \geq \binom{n+d}{d}\cancel{-1} \iff k \geq \left\lceil \frac{\binom{n+d}{d}}{n+1} \right\rceil$$

__Teorema:__ Si $k \geq \left\lceil \frac{\binom{n+d}{d}}{n+1} \right\rceil$ entonces

$$\sigma_k(\nu_d(\mathbb{P}^n)) = \mathbb{P}(\mathrm{Sym}^d(V)).$$

__Teorema:__

$\exists$ abierto de Zariski $\emptyset \neq U \subseteq \mathrm{Sym}^d(V)$ tal que $\forall p \in U$ $\left[\operatorname{rank}_W(p) = \left\lceil \frac{\binom{n+d}{d}}{n+1} \right\rceil\right]$

r

$\left[\forall p \in U\ \exists\ \ell_1, \ldots, \ell_r\right.$ formas lineales en

$$\ell_j = \sum_{i=0}^{n} a_i^{(j)} x_i \ :\ p = \ell_1^d + \cdots + \ell_r^d$$

PREGUNTAS NATURALES: (1) Dado p concreto cómo encontrar los ℓ_i?

(2) Sabemos que hay finitas descomposiciones (cuántas?)

Hoy: APOLARIDAD.

$n=1,\ d=2$ $\quad \underline{\text{rank-W}(p)} \leq \left\lceil \frac{\binom{3}{2}}{2} \right\rceil = \left\lceil \frac{3}{2} \right\rceil = 2$

si p genérico $\quad \mathbb{C}$

Ejemplo:

$$xy = \frac{1}{4}\left[(x+y)^2 - (x-y)^2 \right]$$

$n=2,\ d=3$ $\quad \text{rank-W}(p) \leq \left\lceil \frac{\binom{5}{3}}{3} \right\rceil = \left\lceil \frac{10}{3} \right\rceil = 4$

$$xyz = \frac{1}{24}\left[(x+y+z)^3 + (x+y-z)^3 + (x-y+z)^3 + (x-y-z)^3 \right]$$

[Cómo relacionar con ???]

Teorema [Carlini, Catalisano, Geramita] $\quad \text{rkW}(x_0 \dots x_n) = \frac{1}{2} 2 \cdot 2 \dots 2 = 2^{n-1}$

$$\text{rk-W}\left(x_0^{d_0} \cdots x_n^{d_n} \right) = \frac{1}{d_0+1} \prod_{i=1}^{n} (d_i+1)$$

$$0 \leq d_0 \leq d_1 \leq \dots \leq d_n$$

Obs: · Si $p(x) = \langle a_1, x\rangle^d + \dots + \langle a_s, x\rangle^d$

$$\frac{\partial p(x)}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\langle a_1, x\rangle^d\right) + \dots + \frac{\partial}{\partial x_j}\left(\langle a_s, x\rangle^d\right)$$

$$\left[= d\langle a_1, x\rangle^{d-1} \left[a_j^{(1)}\right] + \dots + d\langle a_s, x\rangle^{d} a_j^{(s)} \right]$$

la derivada de $\langle a, x\rangle^d$ es otra potencia multiplicada por el j-ésimo coeficiente.

$$\left(\sum b_i \frac{\partial}{\partial x_1} \frac{\partial}{\partial x_2^2} \frac{\partial}{\partial x_3} \right) \left[\cdot \langle \vec{a}, \vec{x}\rangle^d \right] = \left[d(d-1)(d-2)(d-3) \langle \vec{a}, \vec{x}\rangle^{d-4} \right. a_1 a_2^2 a_3$$

$$\frac{d!}{(d-4)!}$$

$$\left[\ \right) \cdot \sum b_i\, a_1 a_2^2 \cdot a_3$$

APOLARIDAD

$T = \mathbb{C}[y_1, \dots, y_n]$ T actúa sobre S por diferenciación.

$S = \mathbb{C}[x_1, \dots, x_n]$

$$y_i \in T, \quad F \in S$$

$$y_i(F) := \frac{\partial F}{\partial x_i}$$

Lema:

Si $g \in T_e$ y $F = \langle \vec{a}, \vec{x} \rangle^d$

$$\boxed{g\left(\langle \vec{a}, \vec{x} \rangle^d\right) = \frac{d!}{(d-e)!} \langle \vec{a}, \vec{x} \rangle^{d-e} g(\vec{a})}$$

En particular, si $g \in T_d$ entonces

$$\boxed{g\left(\langle \vec{a}, \vec{x} \rangle^d\right) = d! \, g(\vec{a})}$$

Consecuencia: Tenemos un isomorfismo entre

T_d y $S_d^* = \mathrm{Hom}_{e.v.}(S_d, \mathbb{C})$

APOLARITY PAIRING

[Lema: φ es un isomorfismo (de e.v.)]

$$T_d \xrightarrow{\varphi} S_d^*$$

$$g \longmapsto (p \longrightarrow g(p))$$

Dem.: Suponga que $\varphi(g)=0$

En particular $\forall \vec{a}$ $\quad g(\langle \vec{a}, \vec{x}\rangle^d) = 0$

$\quad \| \quad d!\, g(\vec{a}) \Rightarrow g(\vec{a}) = 0 \quad \forall \vec{a}$

$\Rightarrow g = 0$ así que φ es 1-1

como $\dim(T_d) = \dim(S_d)$ φ es iso ✓.

F

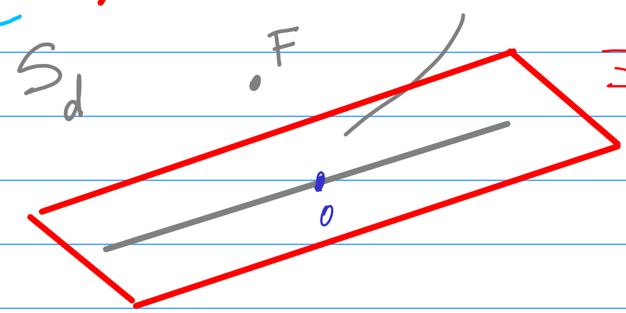
$\|$

$xy = \frac{1}{4}\left[(x+y)^2 + (x-y)^2\right]$

$xy = \frac{1}{4}\left[\langle (1,1), \vec{x}\rangle^2 + \langle (1,-1), \vec{x}\rangle^2\right]$ $\qquad \vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}$

$\left[F \overset{?}{\cancel{\in}} \operatorname{span}\left\{ \langle (1,1), \vec{x}\rangle^2, \langle (1,-1), \vec{x}\rangle^2 \right\}\right]$ APOLARIDAD

(1)



$\exists\ H \in S_d^*$:

(1) $H(\langle (1,1), \vec{x}\rangle^d) = 0$

$H(\langle (1,-1), \vec{x}\rangle^d) = 0$

(2) $H(F) \neq 0$

(2) Usando apolaridad $\exists\ H \in T_d$

(1) $H(\langle (1,1), x\rangle^d) = 0 = d!\, H(1,1)$ $\cdot \left[\begin{array}{l} H(1,1) = 0 \\ H(1,-1) = 0 \end{array}\right]$ $\subseteq \mathbb{P}(S_1)$

PAIRING

(2) $H(F) \neq 0$ $\qquad \Gamma = \{[a^{(1)}], \ldots, [a^{(n)}]\}$

$F \notin \operatorname{span}\left\{ \langle \vec{a}^{(1)}, \vec{x}\rangle^d, \ldots, \langle a^{(n)}, \vec{x}\rangle^d \right\}$

$\Leftrightarrow$ ✓ $\exists\ H \in \left(\mathcal{I}(\{\vec{a}^{(1)}, \ldots, a^{(n)}\})\right)_d$

$H(F) \neq 0 \quad \Leftrightarrow \quad \mathcal{I}(\Gamma) \not\subseteq F^{\perp}$

$F^{\perp} = \{ g \in T^d : g(F) = 0 \}$

FÁCIL de calcular.

Teorema de APOLARIDAD:

$\Gamma \subseteq \mathbb{P}(S_1)$ soporta una expresión de Waring para F

conjunto finito $\Longleftrightarrow$ $\mathcal{I}(\Gamma) \subseteq F^{\perp}$

Obs: Si $\Gamma \subseteq \mathbb{P}(S_1)$

$$\mathcal{I}(\Gamma) \subseteq F^{\perp} \overset{\Rightarrow \checkmark}{\Longleftrightarrow} \left(\mathcal{I}(\Gamma)\right)_d \subseteq F^{\perp}$$

"$\Leftarrow$" $g \in \mathcal{I}(\Gamma)$, $\deg(g) = e$

$e > d \Rightarrow g(F) = 0 \Rightarrow g \in F^{\perp}$ ✓

$e = d \Rightarrow$ hipótesis ✓

$e < d \Rightarrow T_{d-e} \cdot g \in \mathcal{I}(\Gamma)_d$

$$\underbrace{T_{d-e} \cdot g}\ \in F^{\perp} \Rightarrow g \in F^{\perp}$$

$$T_{d-e} \cdot [g(F)] \in S_{d-e}$$

aniquilado por todo elemento de

$$S_{d-e}^{*} = T_{d-e} \Rightarrow g(F) = 0$$

Ejemplo: $F = x_1 x_2 \in S_2$

$$F^{\perp} = \{ g \in T = \mathbb{C}[y_1, y_2] : g(F) = 0 \}$$

$$F^{\perp} \supseteq (y_1, y_2)^3$$

$(F^{\perp})_2$: $\underline{Ay_1^2} + By_1y_2 + \underline{Cy_2^2}\,(x_1 x_2) = 0$

$$B \frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2}(x_1 x_2) = B = 0$$

A, C arbitrarias

$$(F^{\perp})_2 = \langle y_1^2, y_2^2 \rangle$$

$(0) = (F^{\perp})_1$: $(Ay_1 + By_2)(x_1 x_2) = Ax_2 + Bx_1 = 0$

$\iff A = B = 0$

$$\boxed{F^{\perp} = \underbrace{(y_1^2, y_2^2) + (y_1, y_2)^3}}$$

$(y_1^2 - y_2^2)$

$\not\!\exists^1$ y_1, y_2

$\underbrace{(y_1^2 - y_2^2)}_{=} \subseteq F^{\perp}$ $\not\supseteq$

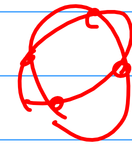
$xy = \frac{1}{4}\left[(x+y)^2 + (x-y)^2\right]$

$(y_1 - y_2)(y_1 + y_2)$

$\{ [1:1], [1:-1] \}$ soporte una exp de long.

Ejemplo: $F(x_1, x_2, x_3) = x_1^a x_2^b x_3^c$

$$F^\perp = \left(y_1^{\overset{a+1}{2}}, y_2^{\overset{b+1}{2}}, y_3^{\overset{c+1}{2}}\right)$$



$$\cup|$$

$$\mathbb{P}^2 \qquad J = \left[\left(y_1^2 - y_2^2, \ y_1^2 - y_3^2\right)\right]$$

$$\cup|$$

$$V(J)$$

$$y_1 \neq 0$$

$$1 - y_2^2 \longrightarrow \boxed{[1 : \pm 1 : \pm 1]}$$

$$1 - y_3^2$$

$$x_1 x_2 x_3 = \frac{1}{24}\Big[(x+y+z)^3 + (x+y-z)^3$$

$$(x-y+z)^3 + (x-y-z)^3\Big]$$

$$\underbrace{x_1 \dots x_n}$$

$$\hookrightarrow \ (y_1^2, \dots, y_n^2)$$

$$(y_1^2 - y_2^2, \dots, y_1^2 - y_n^2)$$

$$\cap| \qquad \leq 2^{n-1} \qquad \text{MACAULAY 2}$$

$$\mathbb{P}^{n-1}$$