Frege’s theorem and his logicism

HIROTOSHI TABATA
Faculty of Education and Regional Sciences, Tottori University, 4-101 Minami, Koyama-cho, Tottori-shi, 680-8551, Japan

Received June 2000

As is well known, Frege gave an explicit definition of number (belonging to some concept) in §68 of his *Die Grundlagen der Arithmetik* (Frege 1884) as follows: the number (die Anzahl) which belongs to the concept \( F \) is the extension of the concept ‘equinumerous to the concept \( F \)’. Here number is defined as an extension of some second-order concept. In other words, a number is a kind of object. After having defined individual numbers, in the following sections (§§74–83) Frege showed how to derive the main theorems of arithmetic. In those derivations, however, Frege did not use the explicit definition of number. Rather he used a kind of contextual definition of number which is now called ‘Hume’s Principle’:

\[
\# F = \# G \iff F \approx G
\]

where ‘ \( \# F \)’ means ‘the number which belongs to the concept \( F \)’, ‘ \( F \approx G \)’ means ‘the concept \( F \) is equinumerous to the concept \( G \)’ or ‘there is a one-to-one correspondence between objects falling under the concept \( F \) and objects falling under the concept \( G \)’. Unlike in Frege’s explicit definition, in this contextual definition the essence of number, that is, what number is, is not given, but only the criterion of identity of numbers.

According to Frege’s later system of *Grundgesetze der Arithmetik* (Frege 1893–1903) the criterion of identity of the extensions of concepts is given by Axiom V as the coextension of those concepts: \( \forall x F x = \forall x G x \iff \forall x(F x \leftrightarrow G x) \) (the extension of the concept \( F \) is the same as the extension of the concept \( G \) if and only if every object falling under the concept \( F \) falls under the concept \( G \) and vice versa). When the explicit definition of number: \( \# F = \# X(F \approx X) \) (the number belonging to the concept ‘equinumerous to the concept \( F \)’) is connected with the instance of (second-order) Axiom V: \( \forall x(F \approx X) = \forall y(G \approx Y) \leftrightarrow \forall \alpha(F \approx H \leftrightarrow G \approx H) \), there is the danger that a contradiction like Russell’s paradox might arise. Therefore, in order to avoid contradiction, it is desirable to use a principle which is guaranteed consistency. Recently several researchers\(^1\) have noticed that in the *Grundlagen*, Frege gave a basis sufficient to derive theorems of arithmetic (including Peano Arithmetic), but using only Hume’s Principle—which is consistent. Boolos believes that this work of Frege is significant enough to deserve the title ‘Frege’s Theorem’.

The purpose of the present paper is, first, to confirm this claim of Boolos’ by following his arguments that Frege’s results, which he calls ‘Frege’s theorem’, can be established, and second, to reconsider the significance of Frege’s results for the logicist program. In section 1, in accordance with Boolos, we construct ‘Frege arithmetic’, a formal system which will serve as the foundation to prove Frege’s theorem, and make the role of Hume’s Principle in this system explicit. Then in section 2, we actually derive some theorems of arithmetic, including Peano’s five axioms, by reconstructing Frege’s arguments in §§74–83 of his *Grundlagen*. Lastly, in section 3, we consider the significance of Frege’s theorem and locate it in his whole logistic programme. This suggests that, despite the discovery of inconsistency in the *Grundgesetze*, Frege’s logicism can be seen in a new and more favourable light.

1. Construction of Frege arithmetic

First, in accordance with Boolos, we construct a formal system which is sufficient for the relevant parts of Frege’s *Die Grundlagen der Arithmetik* (GLA) (that is, the parts of programme deriving the main theorems of arithmetic, developed in §§68–83 of GLA). This formal system is called ‘Frege Arithmetic’ (abbreviated ‘FA’ in what follows). There are several ways to construct FA; here we adopt the way taken in Boolos (1987).

The logical system which is the foundation of FA is standard (binary) second-order

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logic in which quantifiers range over predicate (concept or relation) variables as well as object variables, written in Peano–Russell logical symbols. We have three kinds of variables:

1. object variables (variables ranging over objects): a, b, c, d, m, n, x, y, z, ...;
2. one-place predicate variables (ranging over first level concepts defined over the objects in the domain): F, G, H, ...;
3. two-place predicate variables (ranging over first level binary relations defined over the objects in the domain): ϕ, ψ, ξ, ...

As the sole non-logical symbol in the language of FA, we introduce a two-place predicate symbol ‘η’. We write

\[ Fηx \]

and read this as ‘the concept F belongs to the extension x’ or ‘the concept F is in the extension x’. In other words, the predicate symbol ‘η’ produces an expression with a truth value (as Bedeutung of a sentence) when it is applied to an ordered pair of concept variable and object variable. This is the means to express Frege’s idea that his extensions are objects and that the concept F corresponds to the extension of the higher level concept ‘equinumerous (gleichzahlig) to the concept F’.

Atomic formulae of FA are

\[ Fx, xφy, \text{ and } Fηx. \]

Identity between objects is defined as follows:

\[ x = y \iff \forall F(Fx \leftrightarrow Fy). \]

(\text{In §65 of GLA Frege refers to Leibniz's definition of identity.})

Axioms and rules of inference in FA are the usual axioms and rules of second-order logic. Especially important axioms are the comprehension axioms:

\text{(i) } \exists F \forall x (Fx \leftrightarrow A(x)) \quad \text{(when ‘F’ doesn’t appear free in ‘A(x)’)}
\text{(ii) } \exists ϕ \forall x \forall y (xφy \leftrightarrow B(x, y)) \quad \text{(when ‘ϕ’ doesn’t appear free in ‘B(x, y)’)}

As the sole non-logical axiom of FA, we introduce the formula called ‘Numbers’2

\[ \exists ! x \forall G (Gηx \leftrightarrow F \approx G) \]

where ‘F \approx G’ means ‘F is equinumerous [gleichzahlig] to G’ and abbreviates the second-order logical formula:

\[ \exists ! x \forall y (Gy \leftrightarrow \forall w (xφw \leftrightarrow w = y)) \land \forall y (Gy \leftrightarrow \exists x (Fx \land \forall v (vφy \leftrightarrow v = x))). \]

In §68 of GLA, Frege defined the number which belongs to the concept F as the extension of the second-order concept ‘equinumeITrouss to the concept F’:

\text{die Anzahl, welche dem Begriffe F zukommt, ist der Umfang des Begriffes \text{‘γλειγηζαηληγ δεμ Βεγριωφφει Φ’}.}

As a matter of course, Frege here assumes the existence of the extension of the concept ‘equinumerous to the concept F’. The axiom of Numbers expresses this assumption.

2 Boolos recognizes ‘Numbers’ as non-logical. (Boolos 1987, 186, in my quotations and references, the page-numbers of Boolos’ papers are from texts in Boolos 1998.) But Frege probably considered this axiom of ‘Numbers’ as logical, because he considered extensions as logical objects.
In his sketch of the development of arithmetic in §74 and the following sections of \textit{GLA}, however, Frege does not use this axiom of Numbers (or its equivalent). Rather he actually uses following principle—Hume’s Principle—which is derived from the axiom of Numbers, but doesn’t explicitly refer to the definition of number (The name ‘Hume’s Principle’ is proposed by Boolos on the basis of the fact that Frege makes reference to Hume in §63 of \textit{GLA}):

\begin{align*}
\text{Hume’s Principle: } \# F = \# G \iff F \approx G,
\end{align*}

where ‘\#’ is a function symbol which produces an object-term when it is added to a concept variable, and we read ‘\#F’ as ‘the number which belongs to the concept F’ or ‘the number of Fs’. Now FA (Frege arithmetic) is expanded by the addition of a definition, since the function symbol ‘\#’ is introduced to FA. This principle of Hume’s claims that the criterion of the number of Fs being identical with the number of Gs, is the equinumerosity between objects belonging to the concepts F and G. This principle doesn’t directly commit Frege to the explicit definition of number as the extension of some higher-order concept, which is an object. Hume’s Principle, of course, assumes the existence of numbers. However, it does not commit Frege to answering the question of what number is, what the essence of number is; it simply gives the criterion of identity between numbers.

We can show that Hume’s Principle is satisfied, and consequently that it is consistent. The proof is given by constructing a model M which satisfies Hume’s Principle. First, we make the set U the domain of M:

\[ U = \{0, 1, 2, 3, \ldots, \aleph_0\}. \]

Then we make the power set of U, \(\mathcal{P}(X: X \subseteq U)\), the range of concept variables, and the power set of \(U^2\), \(\mathcal{P}(X: X \subseteq U \times U)\), the range of (two-place) relation variables. U has the significant property that, when we take any cardinal number of any subset of U, U makes the number itself an element of U again. (The set of all natural numbers \(N = \{0, 1, 2, 3, \ldots\}\) lacks this property.) In model M, we interpret the function symbol ‘\#’ as the function \(2^U \rightarrow \{\{V: V \in 2^U\}\}\), that is, the function which produces the cardinal number \(|V|\) of V as output when given any subset V of U as input.

In this model M, we specify the valuation function s in M which gives each expression its semantic value; s gives each object variable an element of U, each concept variable an element of the power set of U, and each relation variable an element of the power set of \(U^2\).

Now it is shown that Hume’s Principle is satisfied by s in M. According to the definition of M and s in M, ‘\#F = \#G’ is satisfied by s in M if and only if the cardinal number of s(F) is identical with the cardinal number of s(G):

\begin{align*}
\text{s(\#F = \#G) = T} & \iff |s(F)| = |s(G)|. \quad (1)
\end{align*}

And further, ‘\(F \approx G\)’ is satisfied by s in M if and only if there exists a one-to-one correspondence between s(F) and s(G):

\begin{align*}
\text{s(F \approx G) = T} & \iff \exists f(s(F) \text{ corresponds one-to-one to } s(G) \text{ by } f) \quad (2)
\end{align*}

But the cardinal number of s(F) is identical with the cardinal number of s(G) if and only if there exists a one-to-one correspondence between them:

\begin{align*}
|s(F)| = |s(G)| & \iff \exists f(s(F) \text{ corresponds one-to-one to } s(G) \text{ by } f) \quad (3)
\end{align*}

\(^3\) Cf. Boolos (1987, 187 f.)
From the above equations 1, 2 and 3, it follows that
\[ s(\# F = \# G) = T \iff s(F \approx G) = T. \]
Therefore, it is given that
\[ s(\# F = \# G \iff F \approx G) = T. \]

This means that every valuation \( s \) satisfies Hume’s Principle in \( M \). Consequently \( M \) is a model of Hume’s Principle.

Likewise we can show that the axiom of Numbers is also satisfied. As domain of \( M \), we take the same set \( U \) as the above, that is, \( U = \{0, 1, 2, 3, \ldots, \aleph_0\} \), and for \( V \subseteq U \) and for \( u \in U \), we put:

\[ V_\eta u \iff (\text{the cardinal number of } V) = u. \]

Then the axiom of Numbers is true in \( M \), that is, Numbers is satisfied by every \( s \) in \( M \).

Since Numbers is true in \( M \), it is consistent. From this consistent axiom of Numbers we can derive Hume’s Principle. The axiom of Numbers claims that, for any given first-order concept \( F \), there exists a unique number \( x \) (the number belonging to \( F \)), which belongs to any concept \( G \) equinumerous to \( F \) as a number \( x \) such that \( F_\eta x \) (from the fact that \( F \approx G \)). Here we define the number belonging to the concept \( F \), \( \# F \), as follows:

**Definition 1** \( \# F = x \iff \forall H (H_\eta x \iff H \approx F) \).

This formula plays a role as the bridge connecting \( \# F \) with the cardinal number \( x \) of \( F \), which is a number such that \( F_\eta x \), and the existence of which is assured, by the axiom of Numbers. Through the medium of this definition, Hume’s Principle is derived from the axiom of Numbers in \( FA \) as follows.

Hume’s Principle: \( \# F = \# G \iff F \approx G \)

First, suppose that \( \# F = \# G \). From the logical truth \( \# F = x \iff \# F = x \), and this supposition, it follows that \( \# F = x \iff \# G = x \). Therefore, from this by definition 1, it follows that

\[ \forall H (H_\eta x \iff H \approx F) \iff \forall H (H_\eta x \iff H \approx G). \] (4)

But from definition 1 it follows that \( \# F = \# F \iff \forall H (H_\eta \# F \iff H \approx F) \). From this and the logical law \( \# F = \# F \), it follows that

\[ \forall H (H_\eta \# F \iff H \approx F). \] (5)

From equations 4 and 5 above, it follows that

\[ \forall H (H_\eta \# F \iff H \approx G). \] (6)

By definition of ‘\( \approx \)’ it follows that \( F \approx F \), therefore from equation 5 it follows that \( F_\eta \approx F \). But from equation 6, \( F_\eta \# F \iff F \approx G \). Therefore from these it follows that \( F \approx G \). Thus it has been shown that \( \# F = \# G \iff F \approx G \).

Next, suppose that \( F \approx G \). Since \( \approx \) is an equivalent relation, for any \( H \), \( H \approx F \iff H \approx G \). Therefore,

\[ \forall H (H_\eta x \iff H \approx F) \iff \forall H (H_\eta x \iff H \approx G). \]

From definition 1, it follows that

\[ \# F = x \iff \forall H (H_\eta x \iff H \approx F), \]
\[ \# G = x \iff \forall H (H_\eta x \iff H \approx G). \]
From these three, it follows that \( \# F = x \leftrightarrow \# G = x \). But \( \# G = \# G \). Therefore, \( \# F = \# G \). Thus, it has been shown that \( F \approx G \rightarrow \# F = \# G \). Q.E.D.

Frege tried to prove Hume’s Principle in §73 of *Die Grundlagen der Arithmetik*. It seems that Frege thought you could freely come and go between ‘the extension of the concept \([H:F \approx H]\) = the extension of the concept \([H:G \approx H]\)’ and ‘\(\forall H(H \approx F \leftrightarrow H \approx G)\)’. The reason why he thought so seems to be that he believed that the fundamental principle which supported Hume’s Principle was the following version of the notorious Axiom V of *Die Grundgesetze der Arithmetik*:

for any second-order concepts \(C, D\)
the extension of \(C\) = the extension of \(D\) ↔ (for any first-order concept \(H, CH \leftrightarrow DH\)).

In fact, since Frege explicitly defined the cardinal number of concept \(F\) as the extension of the second-order concept ‘equinumerous to \(F\)’, he had to regard this principle as fundamental. However, as Boolos points out, a contradiction follows the above principle (but not from the Axiom of Numbers).\(^4\) Boolos’ derivation is as follows. Let \(\eta\) be defined as:

\[
\begin{align*}
F\eta x & \leftrightarrow \text{for some second-order concept } D, \ x = D^* \wedge DF \\
\end{align*}
\]

where \(D^*\) means ‘the extension of \(D\)’. By the comprehension axiom higher by one order \(\exists C \forall F(CF \leftrightarrow \ldots F \ldots)\), we put \(C = [F: \exists x (\neg F\eta x \wedge Fx)]\). And by the lowest-level comprehension axiom \(\exists X \forall x (Xx \leftrightarrow \ldots x \ldots)\), we put \(X = [x: x = C^*]\). Now let \(X\eta C^*\) be supposed. Then, by definition of \(\eta\), there exists some second-order concept \(D\) such that \(C^* = D^* \wedge DX\). From this, by the principle concerned: \(C^* = D^* \leftrightarrow \forall H(CH \leftrightarrow DH)\), it follows that \(CX\). But, by the definition of \(C\), \(\exists x (\neg X\eta x \wedge Xx)\). That is, for some \(x\), \(\neg X\eta x \wedge Xx\). From \(Xx\) by the definition \(X\), it follows that \(x = C^*\). Therefore \(\neg X\eta C^*\). This contradicts the supposition \(X\eta C^*\). Thus we have arrived at a contradiction. So we suppose \(\neg X\eta C^*\). By the definition of \(\eta\), \(\neg \exists D(C^* = D^* \wedge DX)\). \(\therefore \forall D(C^* = D^* \rightarrow \neg DX)\). \(\therefore C^* = C^* \rightarrow \neg CX\). But \(C^* = C^*\). \(\therefore \neg CX\). Then by the definition of \(C\), \(\exists x (\neg X\eta x \wedge Xx)\). \(\therefore \forall X(Xx \rightarrow X\eta x)\). \(\therefore XC^* \rightarrow X\eta C^*\). But, from \(C^* = C^*\) by the definition of \(X\), \(X = [x: x = C^*]\), it follows that \(XC^*\). Therefore \(X\eta C^*\). This contradicts the supposition \(\neg X\eta C^*\). Thus, we have arrived at a contradiction again.

So there is the great significance in the fact that Hume’s Principle can be derived from the axiom of Numbers, because Numbers is free from contradiction. Surprisingly, many theorems of arithmetic can be derivable from Hume’s Principle, even though it appears to be very simple. In other words, Hume’s Principle is consistent, but it is not so weak. In fact, we can derive the axiom of Numbers from Hume’s Principle as follows.

As a bridge connecting Hume’s Principle with the axiom of Numbers, we define ‘\(\eta\)’ as \(F\eta x \leftrightarrow \# F = x\). We must derive the axiom of Numbers \(\forall F \exists x \forall H(H\eta x \leftrightarrow H \approx F)\) from Hume’s Principle \(\# F = \# G \leftrightarrow F \approx G\). Take any first-order concept \(F\). Then we have to show that there exists a unique \(x\) such that \(\forall H(H\eta x \leftrightarrow H \approx F)\). Now take any first-order concept \(H\). From Hume’s Principle, \(\# H = \# F \leftrightarrow H \approx F\), by universal generalization it follows that \(\forall H(\# H = \# F \rightarrow H \approx F)\). As ‘\(\# F\)’ stands for an object, by existential generalization for objects, \(\exists x \forall H(\# H = x \leftrightarrow H \approx F)\). The uniqueness of \(x\) is shown as follows. We put \(\forall H(\# H = x_1 \leftrightarrow H \approx F), \forall H(\# H = x_2 \leftrightarrow H \approx F)\). Now \(F \approx F\), \(\therefore \# F = x_1\) and \(\# F = x_2\). \(\therefore x_1 = x_2\). \(\therefore \exists x \forall H(\# H = x \leftrightarrow H \approx F)\). By the

\(^4\) In Boolos (1986–7, 173 f.)
above definition of ‘η’, \( \exists ! x \forall H (H \eta x \leftrightarrow H \approx F) \). Since F is any first-order concept, it follows that \( \forall F \exists ! x \forall H (H \eta x \leftrightarrow H \approx F) \). Q. E. D.

This result of Frege’s—that many theorems of arithmetic can be derived from Hume’s Principle, which looks simple at first sight, but is really quite strong—is truly worthy of attention. In fact, in \textit{GLA} Frege showed in outline that some main theorems of arithmetic can be derived from Hume’s Principle. In the next section, we will confirm that many theorems of arithmetic, notably the five axioms of Peano Arithmetic, can be derived in Frege’s way.

2. Derivation of main theorems of arithmetic

In this section we derive some main theorems of arithmetic from Hume’s Principle in accordance with Frege’s (1884) programme of \textit{Die Grundlagen der Arithmetik} §§68–83. As the foundation of our derivation, we adopt the second-order logic used in part III of \textit{Begriffsschrift} (Frege 1879). Among the derived theorems are following five axioms of Peano Arithmetic.\(^5\)

1. Zero is a natural number.
2. Every natural number has a unique successor, which is also a natural number.
3. Zero is not the successor of any natural number.
4. For any natural number \( x, y \), if the successor of \( x \) is the same as the successor of \( y \), then \( x \) is the same as \( y \).
5. For any property \( F \), if zero has \( F \), and every successor of a natural number which has \( F \) also has \( F \), then every natural number has the property \( F \) (mathematical induction).

Among notions which appear in these axioms, usually ‘zero’, ‘natural number’, and ‘successor’ are regarded as primitive non-logical notions. But we shall define these notions, in accordance with Frege, by using the most suitable ‘logical’ concepts. In what follows, we derive all the above axioms as theorems in our system FA (Frege Arithmetic). Before we begin the derivations, we rewrite the above five axioms in our (Frege’s) symbols. ‘Num’ in our words\(^6\) means ‘natural number’, ‘xPy’ means ‘\( x \) precedes \( y \)’ or ‘\( y \) is successor of \( x \)’. Then Peano’s five axioms are expressed as follows. (On the right side we indicate whence each Peano axiom is derived in our derivations.)

\[
(1') \text{Num } 0. \quad \text{theorem 8} \\
(2') \forall x (\text{Num } x \rightarrow \exists y (\text{Num } y \wedge xPy \wedge \forall z (xPz \rightarrow z = y))). \quad \text{corollary of theorem 15} \\
(3') \forall x (\text{Num } x \rightarrow \neg xP0). \quad \text{corollary of theorem 3} \\
(4') \forall y \forall z (\text{Num } x \wedge \text{Num } y \wedge \text{Num } z \wedge xPz \wedge yPz \rightarrow x = y) \quad \text{corollary of theorem 2} \\
(5') \forall F [\{F0 \wedge \forall x Fx \wedge xPy \rightarrow Fy\} \rightarrow \forall x (\text{Num } x \rightarrow Fx)] \quad \text{theorem 9}
\]

5 For the expression and the order of presentation of five axioms of Peano Arithmetic, we follow Landau (1951), which develops the number system covering Peano Arithmetic of natural numbers, as the foundation, and of fractions, real numbers, and complex numbers.

Frege distinguished ‘Zahl’ from ‘Anzahl’ in \textit{GLA} §18. He uses ‘Zahl’ to refer to individual numbers, for example, the number 1, the number 2, etc. When he mentions the general concept of number, he uses ‘Anzahl’. Thus ‘Anzahl’ is used as a concept word when he defines individual numbers as ‘0 is the number which belongs to the concept “not identical with itself” (‘0 ist die Anzahl, welche dem Begriffe “sich selbst ungleich” zukommt’). Austin translates ‘Anzahl’ as ‘Number’. We use the symbol ‘Num’ as concept word (or predicate) to stand for the concept under which individual natural numbers or finite numbers (Frege’s ‘endliche Anzahlen’) fall. We define natural number (finite cardinal number) in definition 6.
Now we begin deriving the theorems of arithmetic along the line of Frege’s GLA. In accordance with Frege’s spirit of strict proof, we will derive each theorem as exactly as possible, in other words, we will make proofs with as few gaps as possible. (We often use different predicate letters from those used in section 1, and in principle omit the outermost universal quantifiers.) First, let us again confirm Hume’s Principle as the contextual definition of numbers:

**Hume’s Principle (HP):** \(# F = \# G \iff F \cong G.\)

**Definition 2** (definition of zero): \(0 = \# [x : x \neq x].\)

In GLA §74, Frege adopted the concept ‘not identical with itself’ as the concept under which no object falls—we express this as \([x : x \neq x]\)—that is, as a concept which is logically contradictory. Frege says, ‘I [Frege] could have used for the definition of nought any other concept under which no object falls. But I have made a point of choosing one which can be proved to be such on purely logical grounds; and for this purpose “not identical with itself” is the most convenient that offers, [...]’ (Austin’s translation). According to Frege it can be shown by pure logic that the concept ‘not identical with itself’ is a concept under which no object falls.

In GLA §75, Frege says that no object falls under any concept to which zero belongs, and conversely that zero is the number which belongs to any concept under which no object falls. We derive this proposition as theorem 1.

**Theorem 1:** \(\# F = 0 \iff \forall x \neg Fx.\)

**Proof.** (i) First we derive the right side from the left side of the theorem. Suppose \(\# F = 0.\) (7)

From equation 7 and the definition of zero: \(0 = \# [x : x \neq x]\) (definition 2), by transitivity of identity \((a = b, b = c, \therefore a = c),\) it follows that \(\# F = \# [x : x \neq x].\) (8)

From Hume’s Principle: \(\# F = \# [x : x \neq x] \iff F \cong [x : x \neq x]\) and equation 8, by propositional logic \((A, A \iff B \therefore B),\) it follows that \(F \cong [x : x \neq x].\) (9)

But \(\forall x \neg(x \neq x);\) in other words, the concept \([x : x \neq x]\) is empty. Then \(F\) is also empty because \(F \cong [x : x \neq x].\) If for any object \(x\) it is true that \(Fx,\) then by the definition of \(\cong\) there exists some relation \(\phi\) and a unique object \(y\) such that \(x \phi y \land y \neq y;\) but this contradicts the logical law \(y = y.\) Therefore it follows that \(\forall x \neg Fx.\) (10)

Since we have derived (10) from the supposition equation 7, by conditionalization it follows that \(\# F = 0 \rightarrow \forall x \neg Fx.\) (11)

(ii) Conversely, suppose \(\forall x \neg Fx.\) (12)

From the logical law \(x = x\) and \(x = x \rightarrow (x \neq x \rightarrow Fx)\) by modus ponens, it follows that \(x \neq x \rightarrow Fx.\) (13)
From the logical law $\neg Fx \rightarrow (Fx \rightarrow x \neq x)$ and $\neg Fx$, which results from applying the rule of universal instantiation ($\forall x Px \rightarrow Py$) to supposition equation 11, it follows that

$$Fx \rightarrow x \neq x.$$  \hspace{1cm} (14)

Thus from equations 13 and 14 by propositional logic, it follows that $Fx \leftrightarrow x \neq x$. Therefore by universal generalization, it follows that

$$\forall x (Fx \leftrightarrow x \neq x).$$  \hspace{1cm} (15)

But generally it can be proved that

$$\forall x (Px \leftrightarrow Qx) \rightarrow (P \approx Q).$$  \hspace{1cm} (16)

(This means that there exists a one-to-one correspondence between objects falling under the two concepts which are coextensive with each other. In fact, equation 16 can be proved as follows. First suppose $\forall x (Px \leftrightarrow Qx)$. Now we put $x \phi y \leftrightarrow x = y$; that is, we take ‘ = ’ as one-to-one correspondence between objects falling under concepts $P$ and $Q$ which are coextensive by supposition. Suppose $Px$. From the first supposition $\forall x (Px \leftrightarrow Qx)$, it follows that $Px \leftrightarrow Qx$. So from these $Px$ and $Px \leftrightarrow Qx$, it follows that $Qx$. Therefore from this and the logical law $x = x$ by propositional logic, $Qx \land x = x$. ‘.’ By the definition of $\phi$, it follows that $Qx \land x \phi x$. For any $z$, if $x \phi z$, then by definition of $\phi$ it follows that $z = x$. ‘ :: $\forall z(x \phi z \rightarrow z = x)$, ‘ :: $Qx \land x \phi x \land \forall z(x \phi z \rightarrow z = x)$. By existential generalization, $\exists y[Qy \land x \phi y \land \forall z(x \phi z \rightarrow z = y)]$. ‘ :: $\exists y(Qy \land x \phi y)$. Since this formula has been derived from the supposition $Px$, by conditionalization and universal generalization it follows that $\forall x (Px \rightarrow \exists y(Qy \land x \phi y))$. Likewise it can be shown that $\forall y(Qy \rightarrow \exists x(Px \land x \phi y))$. Therefore $\exists \phi[\forall x (Px \rightarrow \exists y(Qy \land x \phi y)) \land \forall y(Qy \rightarrow \exists x(Px \land x \phi y))]$, that is, $P \approx Q$.)

Therefore, from equation 15 and $\forall x (Fx \leftrightarrow x \neq x) \rightarrow F \approx [x : x \neq x]$, a instance of equation 16, it follows that

$$F \approx [x : x \neq x].$$  \hspace{1cm} (17)

From Hume’s Principle: $\# F = \# [x : x \neq x] \leftrightarrow F \approx [x : x \neq x]$ and equation 17, by propositional logic, it follows that

$$\# F = \# [x : x \neq x].$$  \hspace{1cm} (18)

From the definition of zero: $0 = \# [x : x \neq x]$ and equation 18, by the law of identity $(a = b, c = b \vdash a = c)$,

$$\# F = 0.$$  \hspace{1cm} (19)

Since equation 19 has been derived from the supposition equation 12, by conditionalization it follows that

$$\forall x \neg Fx \rightarrow \# F = 0.$$  \hspace{1cm} (20)

Therefore from equations 11 and 20 by propositional logic it follows that

$$\# F = 0 \leftrightarrow \forall x \neg Fx.$$
So we define the predecessor relation (the converse of successor relation) as follows:

**Definition 3** (definition of predecessor (the converse of successor) relation):

\[ mPn \iff \exists F \exists y (Fy \land n = \# F \land m = \# [x : Fx \land x \neq y]). \]

We read ‘mPn’ as ‘m immediately precedes n’ or ‘m is a predecessor of n’ (‘n follows immediately after m’ or ‘n is a successor of m’).

From definition 3, we get the following theorem, which says that the predecessor relation is a one-to-one correspondence.

**Theorem 2:** \[ mPn \land m'Pn' \rightarrow (m = m' \leftrightarrow n = n'). \]

**Proof.** Suppose

\[ mPn \land m'Pn'. \]  
(21)

By this supposition and the definition of the ‘predecessor relation’ (definition 3), there exist a pair of some concept \( F \) and some object \( y \) and another pair of concept \( F' \) and object \( y' \) such that

\[ Fy \land n = \# F \land m = \# [x : Fx \land x \neq y] \land F'y' \land n' = \# F' \land m' = \# [x' : F'x' \land x' \neq y']. \]  
(22)

(i) Suppose \[ m = m'. \]  
(23)

From equations 22 and 23 by the law of identity, it follows that

\[ \# [x : Fx \land x \neq y] = \# [x' : F'x' \land x' \neq y']. \]  
(24)

So by HP (Hume’s Principle), between objects falling under the concept \( [x : Fx \land x \neq y] \) and the concept \( [x' : F'x' \land x' \neq y'] \) there exists a one-to-one mapping \( \phi \), by which it follows that

\[ [x : Fx \land x \neq y] \approx [x' : F'x' \land x' \neq y']. \]  
(25)

From equation 23 it follows that \( Fy \) and \( F'y' \). So from equation 25 by the mapping \( \phi \cup \{y, y'\} \) which is made from \( \phi \), it follows that

\[ F \approx F'. \]  
(26)

From equation 26, by HP, \( \# F = \# F' \). But from equation 22, it follows that \( \# F = n \) and that \( \# F' = n' \). Therefore

\[ n = n'. \]  
(27)

Since equation 27 has been derived from the supposition equation 23, by conditionalization it follows that

\[ m = m' \rightarrow n = n'. \]  
(28)

(ii) Conversely, suppose

\[ n = n'. \]  
(29)

Then, from equations 22 and 29, it follows that

\[ \# F = \# F'. \]  
(30)

From equation 30, by HP, it follows that

\[ F \approx F'. \]  
(31)
From equation 31, there exists a one-to-one mapping \( \psi \), by which, to \( y' \), there corresponds a unique object \( x \) such that \( Fx \) and \( x\psi y' \), and to \( y \) such that \( Fy \), there corresponds a unique object \( x' \) such that \( F'x' \) and \( y'x' \). Therefore we can make a new mapping \( \phi \) by putting:

\[
\phi = ((\psi - \{ \langle x, y' \rangle, \langle y, x' \rangle \}) \cup \{ \langle x, x' \rangle \}) - \{ \langle y, y' \rangle \}.
\]

By this mapping \( \phi \), we can show that

\[
[x : Fx \land x \neq y] \simeq [x' : F'x' \land x' \neq y'].
\]

From equation 33, by HP,

\[
\#[x : Fx \land x \neq y] = \#[x' : F'x' \land x' \neq y'].
\]

Therefore, from equation 34 and 22 by the law of identity, it follows that

\[
m = m'.
\]

Since (35) has been derived from the supposition equation 29, by conditionalization it follows that

\[
n = n' \implies m = m'.
\]

From equations 28 and 36 by propositional logic,

\[
m = m' \iff n = n'.
\]

Since (37) has been derived from supposition (21), by conditionalization it follows that

\[
mPn \land m'Pn' \implies (m = m' \iff n = n').
\]

As a corollary of theorem 3, we can derive Peano’s fourth axiom, although we have not yet defined the notion of finite cardinal number (natural number) \( 'Num' \).

**Corollary 1** (Peano’s fourth axiom): \( \forall x \forall y \forall z \) (\( \text{Num} x \land \text{Num} y \land \text{Num} z \land xPz \land yPz \implies x = y \)).

**Proof.** From theorem 2 by propositional logic \( (A \implies (B \iff C) \implies (A \land C \implies B) \), it follows that

\[
mPn \land m'Pn' \land n = n' \implies m = m'.
\]

From equation 38, by universal generalization and universal instantiation, it follows that

\[
\forall w(xPz \land yPw \land z = w \implies x = y).
\]
From equation 39, by predicate logic \((\forall w(Qw \rightarrow A) \leftrightarrow (\exists wQw \rightarrow A))\) and propositional logic, it follows that
\[
\exists w(xPz \land yPw \land z = w) \rightarrow x = y. \tag{40}
\]
From equation 40, by predicate logic \((\exists w(xPz \land yPw \land z = w) \leftrightarrow xPz \land yPz)\) and propositional logic, it follows that
\[
xPz \land yPz \rightarrow x = y. \tag{41}
\]
From (41), propositional logic \((A \rightarrow B \cdot C \land A \rightarrow B)\), it follows that
\[
\text{Numx} \land \text{Numy} \land \text{Numz} \land xPz \land yPz \rightarrow x = y. \tag{42}
\]
From equation 42, by universal generalization, it follows that
\[
\forall x \forall y \forall z(\text{Numx} \land \text{Numy} \land \text{Numz} \land xPz \land yPz \rightarrow x = y).
\]

**Theorem 3** (There is no predecessor of zero): \(\neg m P 0\).

**Proof.** We prove this theorem by reductio ad absurdum. Suppose
\[
m P 0. \tag{43}
\]
From equation 43 by the definition of the predecessor relation ‘P’ (definition 3), there exists some concept \(F\) and object \(y\) such that
\[
Fy \land 0 = \# F \land m = [x : Fx \land x \neq y]. \tag{44}
\]
But from theorem 1, \(\# F = 0 \leftrightarrow \forall x \neg Fx\). So from this and \(0 = \# F\) of equation 44, it follows that
\[
\forall \neg Fx. \tag{45}
\]
From equation 45, by universal instantiation it follows that
\[
\neg Fy. \tag{46}
\]
But from equation 44 by propositional logic it follows that
\[
Fy. \tag{47}
\]
equations 46 and 47 contradict each other. Thus, by reductio ad absurdum, it must be that
\[
\neg m P 0.
\]

We can derive Peano’s third axiom as a corollary of theorem 3, whatever ‘Num’ may mean.

**Corollary 2** (Peano’s third axiom): \(\forall x(\text{Numx} \rightarrow \neg x P 0)\).

**Proof.** From the theorem 3: \(\neg m P 0\), by propositional logic \((A \cdot B \rightarrow A)\), it follows that
\[
\text{Numm} \rightarrow \neg m P 0. \tag{48}
\]
From equation 48, by universal generalization it follows that
\[
\forall x(\text{Numx} \rightarrow \neg x P 0).
\]

**Definition 4** (definition of ‘the proper ancestral of a relation’):
\[
xR^* y \iff \forall F[\forall a \forall b((a = x \lor Fa) \land aRb \rightarrow Fb) \rightarrow Fy].
\]
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Here ‘R’ is any binary relation. In order to understand the meaning of the relation ‘R∗’, as a metaphor, we regard ‘R’ as the relation of ‘parent-to-child’, and ‘R∗’ as the relation of ‘ancestor-to-descendant’ (This metaphor originates from Frege’s *Begriffsschrift* part III). Now the formula ∀a∀b((x = a ∨ Fa) ∧ aRb → Fb) is equivalent to ∀b(xRb → Fb) ∧ ∀a∀b(Fa ∧ aRb → Fb). By the above metaphor, ‘∀b(xRb → Fb)’ means ‘every child of x has the property F’, and ‘∀a∀b(Fa ∧ aRb → Fb)’ means ‘for any pair of a parent a and his (her) child b, if the parent has the property F, the child also has F’, that is, ‘the property F is one which is hereditarily passed from a parent to his child’. Therefore, the meaning of the above definition 4 is that ‘x is an ancestor of y if and only if y has any property which every child of x has and which is hereditarily passed from a parent to his child’. We can transfer these ‘parent’(R) and ‘ancestor’(R∗) metaphors by family sequence into the concept of ‘predecessor (successor)’(P) and ‘following’(P∗) in the natural number sequence. But for a while, we think of R and R∗ generally rather than P and P∗. (Frege derived many theorems concerning R∗—he called R∗ the relation of ‘following in a R-sequence’—in the general theory of sequences of part III of his *Begriffsschrift*.)

Now we prepare a method which will be convenient means in the following proofs of theorems concerning ‘the proper ancestral of a relation’ (definition 4).

(♦)METHOD:

In order to prove a theorem which has a form of ‘xR∗y → ... y ...’, put F = [z: ... z ...], and for any object a, b, first suppose that (a = x ∨ Fa) and aRb, then derive Fb.

The reason why this is a convenient method of proof is as follows. Since we have to show that xR∗y → ... y ..., first we shall suppose that xR∗y. In other words, by the definition of R∗, we suppose that, if we take any F which satisfies the condition ($)∀a∀b((a = x ∨ Fa) ∧ aRb → Fb), then it follows that Fy. As we saw in the definition 4, this condition ($) means that property F belongs to every child of x and is hereditary (in respect to the ‘parent-to-child’ relation). So, as F is any property, if we put F = [z: ... z ...] and show that this F satisfies the condition ($), then ‘... y ...’ is derived because Fy is now ‘... y ...’ and just this ‘Fy’ has to be derived. In what follows, we will use this way of proof and will cite it as ‘METHOD (♦)’. (We can regard ‘METHOD(♦)’ a variation of mathematical induction.) Furthermore, also when we apply the ‘proper ancestral’ relation R∗ on the general sequence to the ‘following’ relation P∗ on the natural number sequence which is product of predecessor relation P, we use METHOD(♦).

**Theorem 4:** xRy → xR∗y.

The meaning of this theorem is that the ‘parent-to-child’ relation is a special case of the ‘ancestor-to-descendant’ relation. Frege made this theorem proposition 91 of his *Begriffsschrift* part III.

**Proof:** Suppose

\[ xRy. \] (49)

Since we have to show xR∗y, by the definition of R∗ (definition 4), for any property F, we suppose

\[ \forall a \forall b((a = x \lor Fa) \land aRb \rightarrow Fb). \] (50)
From equation 50, by general instantiation (put a = x, b = y), it follows that

\[(x = x \lor Fx) \land xRy \rightarrow Fy.\] (51)

But, from the logical law \(x = x\) by propositional logic (A,\(\sim\)A \lor B), it follows that

\[x = x \lor Fx.\] (52)

From equations 52 and 49 by propositional logic (A,\(\sim\)A \land B), it follows that

\[(x = x \lor Fx) \land xRy.\] (53)

Consequently, from equations 51 and 53, by propositional logic (modus ponens), it follows that

\[Fy.\] (54)

Since equation 54 has been derived from the supposition equation 50, by conditionalization and universal generalization it follows that

\[\forall F[\forall a \forall b\{(a = x \lor Fa) \land aRb \rightarrow Fb\} \rightarrow Fy].\] (55)

Equation 55 is \(xR^*y\). Since \(xR^*y\) has been derived from the supposition equation 49, it follows that

\[xRy \rightarrow xR^*y.\]

Frege mentioned the transitivity of \(R^*\) as proposition 98 in the *Begriffsschrift*. We also register it as theorem 5.

**Theorem 5** (transitivity of \(R^*\)): \(xR^*y \land yR^*z \rightarrow xR^*z\).

**Proof.** Suppose

\[xR^*y,\] (56)
\[yR^*z,\] (57)
\[\forall a \forall b\{(a = x \lor Fa) \land aRb \rightarrow Fb\}.\] (58)

We have to derive \(Fz\). By the supposition equation 57: \(yR^*z\), that is, \(\forall F[\forall a \forall b\{(a = y \lor Fa) \land aRb \rightarrow Fb\} \rightarrow Fz]\), it is sufficient to show that \(\forall a \forall b\{(a = y \lor Fa) \land aRb \rightarrow Fb\}\), since \(Fz\) is derived from it and supposition equation 57. So for any object a, b, we suppose

\[(a = y \lor Fa) \land aRb.\] (59)

We have to derive \(Fb\). From the supposition equation 56,

\[\forall a \forall b\{(a = x \lor Fa) \land aRb \rightarrow Fb\} \rightarrow Fy.\] (60)

From equations 58 and 60, it follows that

\[Fy.\] (61)

Now from equation 59, it follows that \(a = y \lor Fa\). In the case \(a = y\), from equation 61, it holds that \(Fa\). In the case \(Fa\), trivially \(Fa\). Therefore, anyway it follows that

\[Fa.\] (62)

Therefore, from equation 62, by proposition logic (B,\(\sim\)A \lor B), it follows that \(a = x \lor Fa\). But from equation 59, \(aRb\). From these, by propositional logic, it follows that

\[(a = x \lor Fa) \land aRb.\] (63)
From equation 58 by universal instantiation (twice), it follows that
\[(a = x \vee Fa) \land aRb \to Fb. \quad (64)\]
From equations 63 and 64, by propositional logic, it follows that
\[Fb. \quad (65)\]
Since equation 65 has been derived from the supposition equation 59, it has been shown that
\[(a = y \vee Fa) \land aRb \to Fb. \quad (66)\]
From equation 66, by universal generalization, it follows that
\[\forall a \forall b ((a = y \vee Fa) \land aRb \to Fb). \quad (67)\]
From supposition equations 57 and 67 by universal instantiation and modus ponens, it follows that
\[Fz. \quad (68)\]
Thus, we have derived equation 68 from the supposition equation 58; we have shown that
\[\forall a \forall b ((a = x \vee Fa) \land aRb \to Fb) \to Fz, \quad (69)\]
that is, \(xR^*z\). Since this last formula (69), \(xR^*z\), has been derived from the remaining two suppositions (56), (57), it has been shown that
\[xR^*y \land yR^*z \to xR^*z. \]

Now we apply \(R^*\), which is the relation of ‘ancestor-to-descendant’, to \(P^*\), which is the relation of ‘following after’ on the natural number sequence.

**Theorem 6:** \(xP^*n \to \exists mmPn \land \forall m [mPn \to (xP^*m \lor x = m)].\)

The meaning of this theorem is as follows. ‘If \(n\) follows after \(x\) in the natural number sequence, then there exists some predecessor of \(n\), and every predecessor of \(n\) belongs to the natural number sequence beginning with \(x\), that is, every predecessor of \(n\) follows after \(x\) or it is identical with \(x\).’

**Proof.** We can regard \(P^*\) as a special case of \(R^*\), and moreover theorem 6 has the form ‘\(xP^*n \to \ldots n \ldots\)’, so we can use METHOD(\(\bigstar\)). In order to use METHOD(\(\bigstar\)), we put
\[F = [z : \exists mmPz \land \forall m (mPz \to [xP^*m \lor x = m])] \quad (70)\]
and we suppose
\[xP^*n. \quad (71)\]
For any object \(a, b\), we suppose
\[(a = x \lor Fa) \land aPb. \quad (72)\]
Then it is sufficient to derive \(Fb\) under this supposition equation 72. \(Fb\), which we have to derive, is, \(\exists mmPb \land \forall m (mPb \to (xP^*m \lor x = m)).\) From equation 72, it is derived that \(aPb\). From this, it follows that
\[\exists mmPb. \quad (73)\]
We have derived the first half of \(Fb\). Next we must derive the second half of \(Fb\). Now, for any object \(m\), suppose
\[mPb. \quad (74)\]
From aPb of equation 72 and mPb of equation 74 and theorem 2,

\[ m = a. \]  

(75)

Considering the disjunctive pair of supposition equation 72, we take up two cases.

(i) In case that \( a = x \). From equation 75, it follows that \( x = m \). \( \therefore xP*m \lor x = m \). Therefore, from the supposition equation 74 by conditionalization and universal generalization, it follows that

\[ \forall m(mPb \to (xP*m \lor x = m)). \]  

(76)

(ii) In case that \( Fa \), that is, \( \exists mmPa \land \forall m(mPa \to [xP*m \lor x = m]) \) from equation 70. From the first half of this formula, there exists some object \( m' \) such that

\[ m'Pa. \]  

(77)

From the second half of \( Fa \): \( \forall m(mPa \to [xP*m \lor x = m]) \) and equation 76, by universal instantiation and modus ponens, it follows that

\[ xP*m' \lor x = m'. \]  

(78)

From \( m'Pa \) of equation 77 and \( m = a \) of equation 75,

\[ m'Pm. \]  

(79)

Considering the disjunctive pair of equation 78, we take two cases. (\( \alpha \)) In case \( xP*m' \). From equation 79 and theorem 4: \( m'Pm \to mP*m \) by modus ponens it follows that

\[ m'P*m. \]  

(80)

From \( xP*m' \) and equation 80 by transitivity of \( P* \) (theorem 5), it follows that

\[ xP*m. \]  

(81)

(\( \beta \)) In case \( x = m' \). Then, from \( m'Pm \) of equation 79 it follows that \( xPm \), by theorem 4, it follows that

\[ xP*m. \]  

(82)

In either case (\( \alpha \), (\( \beta \)), it has been derived that \( xP*m \). Therefore it follows that

\[ xP*m \lor x = m. \]  

(83)

Since equation 83 has been derived from the supposition equation 74, by conditionalization and universal generalization it follows that

\[ \forall m(mPb \to (xP*m \lor x = m)). \]  

(84)

Thus, in either case of (i), (ii), we have derived the second half of \( Fb \), that is,

\[ \forall m(mPb \to (xP*m \lor x = m)). \]  

(85)

Therefore from (73) and (85) by propositional logic (\( A, B \vdash A \land B \)), it follows that

\[ \exists mmPb \land \forall (mPb \to (xP*m \lor x = m)), \]

that is, \( Fb \).

Frege mentions in §83 of GLA that no natural number which belongs to the sequence of natural numbers beginning with 0 can follow after itself in the sequence of natural numbers. We derive this proposition theorem 7.

**Theorem 7:** \( 0P*n \to \neg mP*n. \)
The meaning of this theorem is that no number which follows after 0 in the sequence of natural numbers follows after itself. (As a special case, this theorem contains that \( \neg 0P*0 \).)

**Proof.** In order to use METHOD(\( \bullet \)), we put

\[
F = [z: \neg zP*z].
\]  
(86)

Furthermore, for any object \( a, b \), suppose

\[
(a = 0 \lor Fa) \land aPb.
\]  
(87)

What we have to show is \( Fb \), that is, from equation 86, \( \neg bP*b \). Now suppose

\[
bP*b.
\]  
(88)

From equation 88 and theorem 6: \( bP*b \rightarrow \exists mmPb \land \forall m[mPb \rightarrow (bP*m \lor b = m)] \), it follows that

\[
\forall m[mPb \rightarrow (bP*m \lor b = m)].
\]  
(89)

From equation 87, by propositional logic

\[
aPb.
\]  
(90)

From equation 89, by universal instantiation, \( aPb \rightarrow bP*a \lor b = a \). From this and equation 90 by modus ponens, it follows that

\[
bP*a \lor b = a.
\]  
(91)

Considering the disjunctive pair of equation 91, we take two cases.

(i) In case that

\[
bP*a.
\]  
(92)

From \( aPb \) of equation 90 and theorem 4: \( aPb \rightarrow aP*b \), by modus ponens it follows that

\[
aP*b.
\]  
(93)

From equations 93 and 92 and the theorem 5: \( aP*b \land bP*a \rightarrow aP*a \), by propositional logic,

\[
aP*a.
\]  
(94)

(ii) In case that

\[
b = a.
\]  
(95)

From the supposition \( bP*b \) of equations 88 and 95, by the logic of identity, it follows that

\[
aP*a.
\]  
(96)

In either case of (i) and (ii), \( aP*a \). That is,

\[
\neg Fa.
\]  
(97)

Therefore, from \( a = 0 \lor Fa \) of equations 87 and 97, by propositional logic (\( A \lor B, \neg B \therefore A \)),

\[
a = 0.
\]  
(98)

From equations 96 and 98 by the logic of identity (\( \ldots a \ldots, a = 0, \therefore \ldots 0 \ldots \)) it follows that

\[
0P*0.
\]  
(99)
But from theorem 6, by propositional logic \((A \rightarrow B \land C \implies A \rightarrow B)\), \(0P*0 \rightarrow \exists mP0\). Therefore from this and 99, by modus ponens, it follows
\[
\exists mP0
\]
which contradicts theorem 3. Therefore it must be that
\[
\neg bP*b.
\]
Thus it has been shown that \(Fb\).

**Definition 5** (definition of the relation of smaller-to-greater):
\[
m \leq n \iff mPn \lor m = n.
\]
The relation of (not properly) ‘smaller-to-greater’ is the relation of ‘following after or identical’ in our words.

**Definition 6** (definition of natural (finite cardinal) number):
\[
\text{Num } n \iff 0 \leq n.
\]
A natural number is a number which is ‘greater than 0 or identical with 0’, that is, a number which is ‘following after 0 or identical with 0’.

**Theorem 8** (Peano’s first axiom): \(\\text{Num } 0\).
From the logical law \(0 = 0\) by propositional logic, it follows that
\[
0P*0 \lor 0 = 0.
\]
From equation 102, by definition 5,
\[
0 \leq 0.
\]
Therefore, by definition 6, 0 is a finite (natural) number, that is, \(\text{Num } 0\).

**Theorem 9** (Mathematical Induction: Peano’s fifth axiom):
\[
\forall F[\{F0 \land \forall x \forall y (Fx \land xPy \rightarrow Fy)\} \rightarrow \forall x (\text{Num } x \rightarrow Fx)].
\]

**Proof.** Take any property \(F\). Suppose
\[
F0
\]
and suppose
\[
\forall x \forall y (Fx \land xPy \rightarrow Fy).
\]
What we have to show is that, for any object \(z\), \(\text{Num } z \rightarrow Fz\). In other words, by definitions 5 and 6, we have to show that \(0P*z \lor 0 = z \rightarrow Fz\). Now for any object \(z\), suppose
\[
0P*z \lor 0 = z.
\]
Considering the disjunctive pair of equation 106, we take two cases.
(i) In case \(0P*z\), it is sufficient to show that \(0P*z \rightarrow Fz\). In order to use METHOD (\(\spadesuit\)), put
\[
F = [z: Fz].
\]
For any \(a, b\), suppose
\[
(a = 0 \lor Fa) \land aPb.
\]
We have to derive $F_b$. From equation 108, by propositional logic

$$a = 0 \lor F_a. \quad (109)$$

Considering the disjunctive pair of equation 109, we take two cases. If $a = 0$, by supposition equation 104, it follows that $F_a$. If $F_a$, trivially it follows that $F_a$. Anyway it follows that

$$F_a. \quad (110)$$

Since from equation 108 it follows that $aPb$, from this and equation 110 by propositional logic it follows that

$$F_a \land aPb. \quad (111)$$

But from equation 105 by universal instantiation (twice),

$$F_a \land aPb \rightarrow F_b. \quad (112)$$

From equations 111 and 112, by propositional logic, it follows that

$$F_b. \quad (113)$$

Therefore by METHOD($\bullet$), it follows that

$$F_z. \quad (114)$$

(ii) In case $0 = z$. From supposition equation 104, by the law of identity, it follows that

$$F_z. \quad (115)$$

Anyway it follows $F_z$. Since this $F_z$ has been derived from the supposition equation 106, it follows that

$$0P^*z \lor 0 = z \rightarrow F_z. \quad (116)$$

From equation 116, by definitions 5 and 6 and universal generalization, it follows that

$$\forall x(Num x \rightarrow F_x). \quad (117)$$

Since equation 117 has been derived from the suppositions equations 104 and 105, by conditionalization and universal generalization, it follows that

$$\forall F_0[\forall x(\forall \forall y(Fx \land xPy \rightarrow Fy)) \rightarrow \forall x(Num x \rightarrow Fx)].$$

**Theorem 10:** $mPn \land 0P^*n \rightarrow \forall x(x \leq m \leftrightarrow x \leq n \land x \neq n)$. The meaning of this theorem is that, if $m$ is the predecessor of $n$ which follows after $0$ in the natural number sequence beginning with $0$, then, every number which is smaller than $m$ or equal to $m$ is properly smaller than $n$, and conversely every number which is properly smaller than $n$ is smaller than $m$ or equal to $m$.

**Proof.** Suppose

$$mPn \quad (118)$$

$$0P^*n. \quad (119)$$

From these suppositions, we have to derive the equivalent, the second half of the theorem. So, first, in order to derive $x \leq m \rightarrow x \leq n \land x \neq n$, suppose

$$x \leq m. \quad (120)$$
From equation 120 by definition 5 it follows that
\[ xP*m \lor x = m. \] (121)

Considering the disjunctive pair of equation 121, we take two cases.

(i) In case that
\[ xP*m. \] (122)

From equation 118 and theorem 4: mPn \(\rightarrow\) mP*n by modus ponens, it follows that
\[ mP*n. \] (123)

Therefore from equations 122 and 123 by the theorem 5 (transitivity of P*), it follows that
\[ xP*n. \] (124)

(ii) In another case, that is,
\[ x = m. \] (125)

From equations 118 and 125 by the logical law of identity, it follows that xPn. Therefore from this and theorem 4: xPn \(\rightarrow\) xP*n, again it follows that
\[ xP*n. \] (126)

In either case, it follows that
\[ xP*n. \] (127)

From equation 127 by propositional logic, it follows that
\[ xP*n \lor x = n. \] (128)

Therefore, from equation 128 by the definition 5, it follows that
\[ x \leq n. \] (129)

Now in order to derive \(x \neq n\), suppose
\[ x = n. \] (130)

From equations 127 and 130 by the law of identity, it follows that
\[ nP*n. \] (131)

But, from equation 119 and theorem 7: 0P*n \(\rightarrow\) \(\neg\)mP*n, by modus ponens, it follows that
\[ \neg mP*n. \] (132)

Thus a contradiction arises. Therefore
\[ x \neq n. \] (133)

From equations 129, 133 by propositional logic, it follows that
\[ x \leq n \land x \neq n. \] (134)

Since equation 134 has been derived from the supposition equation 120, by conditionalization it follows that
\[ x \leq m \rightarrow x \leq n \land x \neq n. \] (135)

Conversely, in order to derive \(x \leq n \land x \neq n \rightarrow x \leq m\) suppose
\[ x \leq n \land x \neq n. \] (136)
From equation 136 it follows that \(x \leq n\). From this by definition 5, it follows that
\[
x P^* n \lor x = n.
\] (137)

But from equation 136 it follows that \(x \neq n\). From this and equation 137 by propositional logic, it follows that
\[
x P^* n.
\] (138)

From theorem 6: \(x P^* n \rightarrow \exists m (P^*_m \land \forall m (P^*_n \rightarrow (x P^*_m \lor x = m)))\) and equation 138, it follows that
\[
\exists m (P^*_m \land \forall m (P^*_n \rightarrow (x P^*_m \lor x = m))).
\] (139)

From equation 139 by propositional logic it follows that
\[
\forall m (P^*_n \rightarrow (x P^*_m \lor x = m)).
\] (140)

From equation 140 by universal instantiation, it follows that
\[
P^*_m \rightarrow x P^*_m \lor x = m.
\] (141)

From equation 141 and supposition equation 118, it follows that
\[
x P^*_m \lor x = m.
\] (142)

From (142) by definition 5, it follows that
\[
x \leq m.
\] (143)

Since equation 143 has been derived from supposition equation 136, by conditionalization, it follows that
\[
x \leq n \land x \neq n \rightarrow x \leq m.
\] (144)

From (135) and (144), by propositional logic and universal generalization, it follows that
\[
\forall x (x \leq m \leftrightarrow x \leq n \land x \neq n).
\] (145)

Since equation 145 has been derived from supposition equations 118 and 119, by conditionalization and propositional logic, it follows that
\[
P^*_m \land \forall P^*_n \rightarrow \forall x (x \leq m \leftrightarrow x \leq n \land x \neq n).
\]

**Theorem 11:** \(P^*_m \land \forall P^*_n \rightarrow \# [x : x \leq m] \rightarrow \# [x : x \leq n] \land x \neq n\).

The meaning of this theorem is as follows: ‘If \(m\) is the predecessor of \(n\) which follows after 0 in the natural number sequence beginning with 0, then, the number which belongs to the concept “smaller than \(m\) or identical with \(m\)” is the predecessor of the number which belongs to the concept “smaller than \(n\) or identical with \(n\)”’.

**Proof.** Suppose
\[
P^*_m \land \forall P^*_n.
\] (146)

From equation 146 and theorem 10 by modus ponens it follows that
\[
\forall x (x \leq m \leftrightarrow x \leq n \land x \neq n).
\] (147)

From equation 147 by the law \(\forall x (P \leftarrow Q) \rightarrow (P \equiv Q)\), which is used in the proof of theorem 1, it follows that
\[
[x : x \leq m] \equiv [x : x \leq n \land x \neq n].
\] (148)
From equation 148, by HP (Hume’s Principle), it follows that
\[
\# [x : x \leq m] = \# [x : x \leq n \land x \neq n].
\] (149)

Now tentatively put
\[
F = [x : x \leq n].
\] (150)

(This tentative definition is eliminated by existential generalization of line equation 157.) From the law of identity: \( n = n \) by propositional logic it follows that
\[
nP^*n \lor n = n.
\] (151)

From equation 151 by definition 5, it follows that
\[
n \leq n.
\] (152)

From equation 152 by tentative definition 151, it follows
\[
F_n.
\] (153)

From equation 150 and the logical law \( \# F = \# F \) by the law of identity \((F = G, \phi(F) \therefore \phi(G))\), it follows that
\[
\# F = \# [x : x \leq n].
\] (154)

From equation 149 by equation 150, it follows that
\[
\# [x : x \leq m] = \# [x : Fx \land x \neq n].
\] (155)

From equations 153, 154 and 155 by propositional logic, it follows that
\[
F_n \land \# [x : x \leq n] = \# F \land \# [x : x \leq m] = \# [x : Fx \land x \neq n].
\] (156)

From equation 156 by existential generalization,
\[
\exists F \exists y (Fy \land \# [x : x \leq n] = \# F \land \# [x : x \leq m] = \# [x : Fx \land x \neq y]).
\] (157)

From equation 157 by definition 3 (definition of P),
\[
\# [x : x \leq m] P \# [x : x \leq n].
\] (158)

Since equation 158 has been derived from supposition equation 146, by conditionalization it follows that
\[
mP n \land 0P^*n \rightarrow \# [x : x \leq m] P \# [x : x \leq n].
\]

Theorems 10 and 11 above play a role of lemma for the next theorem (theorem 12) which Frege explicitly mentions in §82 of GLA.

**Theorem 12:** \( mP n \rightarrow (0 \leq m \land mP \# [x : x \leq m] \rightarrow 0 \leq n \land nP \# [x : x \leq n]). \)

**Proof.** Suppose
\[
mP n
\] (159)
\[
0 \leq m \land mP \# [x : x \leq m].
\] (160)

From equation 160 by propositional logic, it follows that
\[
0 \leq m.
\] (161)

From equation 161 by definition 5, it follows that
\[
0P^*m \lor 0 = m.
\] (162)
(i) In case that 0P*m. (163) From equation 159 and theorem 4: mPn → mP*n, it follows that mP*n. (164) From equations 163 and 164 by theorem 5 (transitivity of P*), it follows that 0P*n. (165)

(ii) In another case that 0 = m. (166) From equations 159 and 166 by the law of identity, it follows that 0Pn. (167) From equation 167 by theorem 4: 0Pn → 0P*n, it follows that 0P*n. (168) Thus in either case, it follows that 0P*n. (169) From this by propositional logic, it follows that 0P*n ∨ 0 = n. (170) From equation 170 by definition 5, it follows that 0 ≤ n. (171) On the other hand, from equation 160 by propositional logic it follows that mP#[x:x ≤ m]. (172) From equations 159 and 172 by propositional logic, it follows that mPn ∧ mP#[x:x ≤ m]. (173) From theorem 2 by substitution, it follows that mPn ∧ mP#[x:x ≤ m] → (m = m ↔ n = # [x:x ≤ m]). (174) From equations 173 and 174 and the logical law m = m by propositional logic, it follows that n = # [x:x ≤ m]. (175) From equations 159 and 169 mPn ∧ 0P*n. (176) From equation 176 and theorem 11: mPn ∧ 0P*n → # [x:x ≤ m]P# [x:x ≤ n] by modus ponens, it follows that # [x:x ≤ m]P# [x:x ≤ n]. (177) Consequently, from equations 175 and 177 by the law of identity, it follows that nP# [x:x ≤ n]. (178) Therefore, from equations 171 and 178 by propositional logic, it follows that 0 ≤ n ∧ nP# [x:x ≤ n]. (179)
Since equation 179 has been derived from suppositions equations 159 and 160 by conditionalization (twice), it follows that
\[ mPn \rightarrow (0 \leq m \land mP \# [x:x \leq m] \rightarrow 0 \leq n \land nP \# [x:x \leq n]). \]

**Theorem 13:** \(0P \# [x:x \leq 0].\)
The meaning of the theorem is, roughly, that the number which belongs to the concept ‘smaller than zero or identical with zero’, that is, number 1 is the successor of zero. (In fact, since only zero falls under the concept \([x:x \leq 0]\), it follows that \(\forall x(x = 0 \leftrightarrow x \leq 0)\). ... \([x:x = 0]\). ... By HP \(\# [x:x = 0] = \# [x:x \leq 0]\). Therefore by Frege’s definition of number 1 (1 = \(\# [x:x = 0]\) GLA §77), it follows that \(\# [x:x \leq 0] = 1\).) But the real meaning of theorem 13 is as follows. In §82 of GLA, Frege says that, what is asserted of \(m\) and \(n\) in theorem 12, holds for the number zero. So this theorem plays a role of preparation of the proof of theorem 15 using Mathematical Induction.

**Proof.** As a tentative definition put
\[ F = [x:x \leq 0]. \] (180)
From the logical law \(0 = 0\) by propositional logic, it follows that
\[ 0P*0 \lor 0 = 0. \] (181)
From equation 181 by definition 5, it follows that
\[ 0 \leq 0. \] (182)
From equations 182 and 180, it follows that
\[ F0. \] (183)
And from equation 180 and the logical law \(# F = \# F\) by the (second-order) law of identity, it follows that
\[ \# F = \# [x:x \leq 0]. \] (184)
Now if \(xP*0\), then, by the first half of theorem 6, it follows that \(\exists m mP0\); but this contradicts theorem 3: \(\neg m P0.\) Therefore it follows that
\[ \neg xP*0. \] (185)
But from the logical law \(x \leq 0 \leftrightarrow x \leq 0\) and definition 5, it follows that
\[ x \leq 0 \leftrightarrow xP*0 \lor x = 0. \] (186)
From equation 186 by propositional logic \((A \leftrightarrow B \vdash A \land C \leftrightarrow B \land C)\), it follows that
\[ x \leq 0 \land x \neq 0 \leftrightarrow (xP*0 \lor x = 0) \land x \neq 0. \] (187)
From propositional logic \(((A \lor B) \land \neg B \leftrightarrow A \land \neg B)\), it follows that
\[ (xP*0 \lor x = 0) \land x \neq 0 \leftrightarrow xP*0 \land x \neq 0. \] (188)
From equations 187 and 188, by propositional logic \((A \leftrightarrow B, B \leftrightarrow C \vdash A \leftrightarrow C)\), it follows that
\[ x \leq 0 \land x \neq 0 \leftrightarrow xP*0 \land x \neq 0. \] (189)
From equation 185 by propositional logic \((A \vdash A \lor B)\), it follows that
\[ \neg xP*0 \lor x = 0. \] (190)
From equation 190 by propositional logic ($\neg A \lor B \vdash \neg(A \land \neg B)$), it follows that
$$
\neg(xP*0 \land x \neq 0).
$$
(191)

Therefore, from equations 189 and 191, by propositional logic ($A \leftrightarrow B, \neg B \vdash \neg A$), it follows that
$$
\neg(x \leq 0 \land x \neq 0).
$$
(192)

From equation 180, it follows that
$$
x \leq 0 \leftrightarrow Fx.
$$
(193)

From equations 192 and 193 by the rule of substitution of equivalents ($A \leftrightarrow B, (\ldots A \ldots) \vdash (\ldots B \ldots)$), it follows that
$$
\neg(Fx \land x \neq 0).
$$
(194)

From equation 194 by universal generalization, it follows that
$$
\forall x \neg(Fx \land x \neq 0).
$$
(195)

But from theorem 1 by substituting ‘Fx \land x \neq 0’ for ‘Fx’ it follows that
$$
\# [x: Fx \land x \neq 0] = 0 \leftrightarrow \forall \neg(Fx \land x \neq 0).
$$
(196)

Therefore, from equations 195 and 196 by propositional logic, it follows that
$$
\# [x: Fx \land x \neq 0] = 0.
$$
(197)

From equations 183 and 184 and 197 by propositional logic, it follows that
$$
F0 \land \# [x: x \leq 0] = \# F \land 0 = \# [x: Fx \land x \neq 0].
$$
(198)

Therefore, from equation 198 by existential generalization, it follows that
$$
\exists F \exists y(Fy \land \# [x: x \leq 0] = \# F \land 0 = \# [x: Fx \land x \neq y]).
$$
(199)

Since equation 199 no longer depends upon the tentative definition 180, from equation 199 by definition 3, it follows that
$$
0P \# [x: x \leq 0].
$$

**Theorem 14:** $0 \leq n \leftrightarrow 0 \leq n \land nP \# [x: x \leq n]$.

**Proof.** Suppose
$$
0 \leq n.
$$
(200)

From equation 200 by definition 5, it follows that
$$
0P*n \lor 0 = n.
$$
(201)

(i) In case that
$$
0 = n.
$$
(202)

From equation 202 and theorem 13: $0P \# [x: x \leq 0]$ by the law of identity, it follows that
$$
nP \# [x: x \leq n].
$$
(203)

From equations 200 and 203 by propositional logic ($A, B \vdash A \land B$), it follows that
$$
0 \leq n \land nP \# [x: x \leq n].
$$
(204)

(ii) In case that
$$
0P*n.
$$
(205)
What we have to show is that $0P\ast n \rightarrow 0 \leq n \land nP \# [x : x \leq n]$. In order to use METHOD(⊙), put
\[ F = [z : 0 \leq z \land zP \# [x : x \leq z]]. \tag{206} \]

For any object $a$, $b$, suppose
\[ (a = 0 \lor Fa) \land aPb. \tag{207} \]

Here it is sufficient to derive $Fb$, that is, $0 \leq b \land bP \# [x : x \leq b]$. From equation 207 by propositional logic ($A \land B \colon \vdash A$), it follows that
\[ a = 0 \lor Fa. \tag{208} \]

Again from equation 207 by propositional logic ($A \land B \colon \vdash B$), it follows that
\[ aPb. \tag{209} \]

Considering the disjunctive pair of equation 208, we suppose two cases. ($\alpha$) In case that
\[ a = 0. \tag{210} \]

From the logical law $0 = 0$ by propositional logic, it follows that
\[ 0P\ast 0 \lor 0 = 0. \tag{211} \]

From equation 211 by definition 5, it follows that
\[ 0 \leq 0. \tag{212} \]

From equations 210 and 212 by the law of identity ($a = b, Fb \colon \vdash Fa$), it follows that
\[ 0 \leq a. \tag{213} \]

Furthermore, from theorem 13: $0P \# [x : x \leq 0]$ and equation 210 by the law of identity, it follows that
\[ aP \# [x : x \leq a]. \tag{214} \]

From equations 213 and 214 by propositional logic ($A, B \colon \vdash A \land B$), it follows that
\[ 0 \leq a \land aP \# [x : x \leq a]. \tag{215} \]

From equations 206 and 215 by the (second-order) law of identity ($F = G, Ga \colon \vdash Fa$), it follows that
\[ Fa. \tag{216} \]

($\beta$) In case that
\[ Fa. \tag{217} \]

Trivially it follows that $Fa$. Therefore in either case it follows that
\[ Fa. \tag{218} \]

From theorem 12: $aPb \rightarrow (0 \leq a \land aP \# [x : x \leq a] \rightarrow 0 \leq b \land bP \# [x : x \leq b])$ and equation 206 by the law of identity (twice),
\[ aPb \rightarrow (Fa \rightarrow Fb). \tag{219} \]

Therefore, from equations 219 and 209 and 218 by modus ponens (twice), it follows that
\[ Fb. \tag{220} \]

Therefore, by METHOD(⊙), it follows that
\[ 0 \leq n \land nP \# [x : x \leq n]. \tag{221} \]
Thus in both cases (i) and (ii), it follows that

$$0 \leq n \land nP \# [x: x \leq n]. \quad (222)$$

Since equation 222 has been derived from supposition equation 200, by conditionalization it follows that

$$0 \leq n \rightarrow 0 \leq n \land nP \# [x: x \leq n].$$

**Theorem 15**: Num $n \rightarrow nP \# [x: x \leq n].$

The meaning of the theorem is that, if $n$ is a natural number, then the number which belongs to the concept ‘smaller than $n$ or identical with $n$’ is a successor of $n$.

**Proof.** Suppose 

Num $n.$ \hspace{1cm} (223)

From equation 223 by definition 6, it follows that

$$0 \leq n. \quad (224)$$

From equation 224 and theorem 14: $0 \leq n \rightarrow 0 \leq n \land nP \# [x: x \leq n]$ by propositional logic $(A, A \rightarrow B \land C \rightarrow C)$, it follows that

$$nP \# [x: x \leq n]. \quad (225)$$

Since equation 225 has been derived from supposition equation 223, by conditionalization, it follows that

$$\text{Num } n \rightarrow nP \# [x: x \leq n].$$

**Proof.** In order to prove theorem 15, we use mathematical induction (theorem 9) without using theorem 14. Put

$$F = [x: 0 \leq z \land z \leq \# [x: x \leq z]]. \quad (226)$$

From theorem 13: $0P \# [x: x \leq 0]$ and $0 \leq 0$—which is derived from $0 = 0$ by propositional logic and definition 5—by propositional logic, we can derive $F0$, that is,

$$0 \leq 0 \land 0 \leq \# [x: x \leq 0]. \quad (227)$$

And by taking ‘$F$’ as one above equation 226, from theorem 12: $xPy \rightarrow (0 \leq x \land xP \# [z: z \leq x] \rightarrow 0 \leq y \land yP \# [z: z \leq y])$ by propositional logic and universal generalization it follows that

$$\forall x \forall y (Fx \land xPy \rightarrow Fy). \quad (228)$$

From $F0$ of equations 227 and 228 by propositional logic, it follows that

$$F0 \land \forall x \forall y (Fx \land xPy \rightarrow Fy). \quad (229)$$

From mathematical induction (theorem 9): $\forall F ([F0 \land \forall x \forall y (Fx \land xPy \rightarrow Fy)] \rightarrow \forall n (\text{Num } n \rightarrow Fn))$, by universal instantiation and modus ponens, we can derive $\forall n (\text{Num } n \rightarrow Fn)$, that is,

$$\forall n (\text{Num } n \rightarrow 0 \leq n \land n \leq \# [x: x \leq n]). \quad (230)$$

Therefore, from equation 230 by universal instantiation and propositional logic $(A \rightarrow B \land C \rightarrow A \rightarrow C)$, it follows that

$$\text{Num } n \rightarrow n \leq \# [x: x \leq n].$$
**Corollary** (Peano’s second axiom): \( \text{Num } m \rightarrow \exists ! n (\text{Num } n \land m \text{Pn}). \)

**Proof.** Suppose

\[ \text{Num } m. \quad (231) \]

From equation 231 and theorem 15: \( \text{Num } m \rightarrow m \text{P}\!\![x: x \leq m] \) by modus ponens, it follows that

\[ m \text{P}\!\![x: x \leq m]. \quad (232) \]

From theorem 4: \( m \text{Pn} \rightarrow m \text{P}\!\![n] \) by substitution, it follows that

\[ m \text{P}\!\![x: x \leq m] \rightarrow m \text{P}\!\![x: x \leq m]. \quad (233) \]

From equations 232 and 233 by modus ponens, it follows that

\[ m \text{P}\!\![x: x \leq m]. \quad (234) \]

From equation 231 by definition 6: \( \text{Num } m \iff 0 \leq m \), it follows that

\[ 0 \leq m. \quad (235) \]

From equation by the definition of ‘\( \leq \)’ (definition 5), it follows that

\[ 0 \text{P}\!\![n] \lor 0 = m. \quad (236) \]

Considering the disjunctive pair of equation 236, we take two cases. (i) In case that

\[ 0 \text{P}\!\![n]. \quad (237) \]

From equations 237 and 234 and theorem 5 (transitivity of ‘\( \text{P}\!\!^* \)' ): \( 0 \text{P}\!\![n] \land m \text{P}\!\![x: x \leq m] \rightarrow 0 \text{P}\!\![x: x \leq m] \), by propositional logic\((A, B, A \land B \rightarrow C, \therefore C)\), it follows that

\[ 0 \text{P}\!\![x: x \leq m]. \quad (238) \]

(ii) In case that

\[ 0 = m. \quad (239) \]

From equations 239 and 234 by the law of identity, it follows that

\[ 0 \text{P}\!\![x: x \leq m]. \quad (240) \]

Thus in either case of (i) and (ii) it follows that

\[ 0 \text{P}\!\![x: x \leq m]. \quad (241) \]

From equation 241 by propositional logic\((A, \therefore A \lor B)\), it follows that

\[ 0 \text{P}\!\![x: x \leq m] \lor 0 = \#[x: x \leq m]. \quad (242) \]

From equation 242 by the definition of ‘\( \leq \)’ (definition 5), it follows that

\[ 0 = \#[x: x \leq m]. \quad (243) \]

From equation 243 by the definition of ‘\( \text{Num} \)’ (definition 6), it follows that

\[ \text{Num } \#[x: x \leq m]. \quad (244) \]

From equations 232 and 244 by propositional logic\((A, B, \therefore A \land B)\), it follows that

\[ \text{Num } \#[x: x \leq m] \land m \text{P}\!\![x: x \leq m]. \quad (245) \]

From equation 245 by existential generalization, it follows that

\[ \exists n (\text{Num } n \land m \text{Pn}). \quad (246) \]
Uniqueness of $n$ in equation 246 is shown as follows. Suppose
\[ mPn \land mPn'. \]  
(247)
But from theorem 2 it follows that
\[ mPn \land mPn' \rightarrow n = n'. \]  
(248)
Consequently, from equations 247 and 248 by modus ponens, it follows that
\[ n = n'. \]  
(249)
Thus we derived the uniqueness of ‘$n$’ in equation 246; so it follows that
\[ \exists n(\text{Num } n \land mPn). \]  
(250)
Since equation 250 has been derived from supposition equation 231, from equation 250 by conditionalization it follows that
\[ \text{Num } m \rightarrow \exists n(\text{Num } n \land mPn). \]
Thus we have shown that, for any natural number (finite number), there exists a unique successor of the number which is also a natural number.

3. The significance of Frege’s results

In this last section, we consider the philosophical and historical significance of Frege’s Theorem. At present we cannot draw any decisive conclusion but can only make several suggestions. We take up three important respects to be argued.

Frege’s theorem and the logicist programme

Frege’s logicist programme—the reduction of mathematics (number theory and analysis) to logic—was carried out in his Grundgesetze der Arithmetik. It seems that as a whole, or in Frege’s original form, his logicist programme ended in failure, because there was a contradiction in the system of Grundgesetze. In this later system, Frege adopted very powerful notions such as the extensions of concepts (classes), and gave explicit definitions of various mathematical notions and objects (including the definition of number). But if we restrict our view to his Grundlagen, Frege’s investigations constitute a modest success; that is, in deriving Peano Arithmetic from Hume’s Principle in second-order logic. As we showed in detail above, many main theorems of arithmetic can be derived from second-order logic with Hume’s Principle. On the one hand, Hume’s Principle:
\[ \# F = \# G \iff F \approx G \]
(the number of Fs is identical with the number of Gs if and only if the concept $F$ is equinumerous to the concept $G$), that is, there is a one-to-one mapping between objects which fall under $F$ and objects which fall under $G$) can be regarded as a contextual definition of number. ‘$\# F$’ means only ‘the number which belongs to the concept $F$’, and we cannot know what number is from Hume’s Principle. We are given only the criterion of identity of numbers. On the other hand, in order to decide the truth value of such a form of sentence as ‘$\# F = x$’, Frege gave the explicit definition of number:
\[ \# F = 'X(F \approx X) \]
(the number of Fs is identical with the extension of the concept ‘equinumerous to the concept $F$’). But the notion of the extensions of concepts (classes) is contained in this
Frege's theorem and his logicism

Therefore the criterion of identity of numbers is given by Axiom V (the general principle which gives the identity condition of the extensions of concepts—and which contains an inconsistency):

\[ X(F \approx X) = Y(G \approx Y) \iff \forall H(F \approx H \iff G \approx X) \]

(the extension of the concept ‘equinumerous to the concept F’ is identical with the extension of the concept ‘equinumerous to the concept G’ if and only if every concept which is equinumerous to the concept F is equinumerous to the concept G and vice versa).

So Frege’s Theorem means the success of logicism, in a modest form. Logicism aims at (i) logical definition of mathematical concepts and (ii) logical (gap-free) derivation of theorems of mathematics. Historically logicism arose with the movement of arithmetization in nineteenth century mathematics. Many mathematicians tried to reduce analysis to arithmetic in order to give mathematics a firm foundation. But arithmetic was left unanalysed. Frege’s motivation was to give arithmetic itself a logical foundation, that is, to reduce arithmetic to logic. Frege’s Theorem shows that the arithmetic of natural numbers can be derived from second-order logic with Hume’s principle, which is free from inconsistency.

Now a very narrow conception of logic is dominant. Quine recommends that we restrict the scope of logic to first-order predicate logic, on at least two grounds: completeness; and concurrence of diverse definitions of logical truth. Quine deplores the use of predicate letters as quantified variables, even when the values are sets, let alone attributes. The reason is his nominalism. ‘Predicates have attributes as their “intensions” or meanings […], and they have sets as their extensions; but they are names of neither. Variables eligible for quantification therefore do not belong in predicate positions. They belong in name positions.’ But such a conception leads to the complete separation of mathematics from logic. For the expressive power of first-order logic is very weak; so it is impossible for first-order logic to logically define the fundamental concepts and derive the fundamental theorems of mathematics. Consequently first-order logic cannot explain the notions of the fundamental parts of mathematics. For example, the notion of the ancestral of a relation cannot be expressed in first-order logic, nor can even the law of identity (‘\( x = x \)’) be derived. Frege showed that second-order logic supplemented by Hume’s Principle yields the arithmetic of natural numbers, the foundational parts of mathematics. The success of Frege’s (modest) logicist programme should suggest a revised, broader conception of logic, and a recovery of the connection of logic and (foundational parts of) mathematics.

Frege’s theorem and the infinity of numbers

We derived Peano’s second axiom (‘every natural number has a unique successor which is also natural number’) as a corollary of theorem 15 (‘If n is a natural number, then, the number which belongs to the concept “smaller than n or identical with n” is the successor of n’). Since the ramification and repetition of the natural number sequence is prohibited by theorem 2, according to which the ‘predecessor-to-successor’ relation is one-to-one, this corollary (Peano’s second axiom) implies that there are infinitely many natural numbers. More specifically, the sequence of natural numbers is given by Axiom V, and the criterion of identity of numbers is given by Axiom V (the general principle which gives the identity condition of the extensions of concepts—and which contains an inconsistency):

\[ X(F \approx X) = Y(G \approx Y) \iff \forall H(F \approx H \iff G \approx X) \]

the extension of the concept ‘equinumerous to the concept F’ is identical with the extension of the concept ‘equinumerous to the concept G’ if and only if every concept which is equinumerous to the concept F is equinumerous to the concept G and vice versa.

So Frege’s Theorem means the success of logicism, in a modest form. Logicism aims at (i) logical definition of mathematical concepts and (ii) logical (gap-free) derivation of theorems of mathematics. Historically logicism arose with the movement of arithmetization in nineteenth century mathematics. Many mathematicians tried to reduce analysis to arithmetic in order to give mathematics a firm foundation. But arithmetic was left unanalysed. Frege’s motivation was to give arithmetic itself a logical foundation, that is, to reduce arithmetic to logic. Frege’s Theorem shows that the arithmetic of natural numbers can be derived from second-order logic with Hume’s principle, which is free from inconsistency.

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8 See Quine (1970, 90–91.)
9 Quine (op. cit., 67.)
numbers \((0, 1, 2, 3, \ldots)\) forms a ‘progression’. A progression is a infinite sequence ‘in which there is a first member, a successor to each member (so that there is no last term), no repetitions, and every member can be reached from the start in a finite number of steps’.\(^{10}\) Whatever numbers may be, Frege can logically derive a progression. In fact, the concepts used in theorem 2 and theorem 15 are all logical in the sense that they are expressed by (object or predicate) variables with logical connectives and (first or second-order) quantifiers. The concept of one-to-one relation can be logically expressed. Furthermore, the concepts ‘finite number (Num)’ and ‘smaller than or identical with’ are reduced to ‘=’ and ‘P*’—following after in the P-sequence’ or generally ‘proper ancestral of a relation’—, which are defined in Frege’s second-order logic using only logical concepts (in the above sense). As a notion which stands for an object, the number operator \([x: \ldots x \ldots]\) is used. But in the process of derivation of theorems, the explicit definition of number is not used. Thus Frege derived the existence of infinitely many objects, natural numbers, from only logical concepts, without commitment to an ontology of extensions.

It has often been pointed out that Russell’s logicism as developed within his type theory brought in non-logical elements by assuming the ‘axiom of infinity’ which postulates ‘the existence of a class consisting of individuals in the world’ [my emphasis].\(^{11}\) Certainly we need infinitely many numbers to develop the arithmetic of natural numbers; but Frege could show how to get infinitely many natural numbers without depending upon such a metaphysical assumption as the axiom of infinity. The ontology which is required by Frege’s derivation of natural numbers is very weak, and it is neutral in the sense that it has only one kind of objects and doesn’t need any hierarchy of objects like that of type theory (individuals, sets of individuals, sets of sets of individuals, etc.).

**Frege’s theorem and contextual definition**

Dummett wrote, ‘[...] it was a tour de force on Frege’s part to combine a vehement advocacy of Platonism with an unreserved logicism about number theory and analysis’.\(^{12}\) When he combined Platonism with ‘unreserved’ logicism in Grundgesetze, Frege abandoned the contextual definition of number and the context principle. He adopted the extensions of concepts (classes) to give an explicit definition of number. In order to make his logicism more powerful than that in Grundlagen (GLA), Frege needed the notion of the extensions of concepts (classes) which gives the strong ontological (metaphysical) foundation for his logicism. Armed with the ontology of extensions and explicit definitions (in Grundgesetze), however, his logicism became too powerful to keep consistency—which is necessary, if modest, logical condition.

\(^{10}\) Russell (1919, p. 8.) In the quotation, I have replaced Russell’s word ‘term’ by ‘member’.

\(^{11}\) According to Russell, ‘the axiom of infinity’ is formulated as follows: ‘If n be any inductive (i.e. finite) cardinal number, there is at least one class of individuals having n terms’ (Russell, 1919, 131). Russell defined numbers as classes of classes. So he thought that, in order to deal with the number n, he needs at least one class consisting of n members which are individuals in the world. For example, he argues as follows. According to him, when we define numbers as classes of classes, it is easy to prove Peano’s axiom 1 and axiom 2. But if the number of individuals in the world is finite, axiom 4 (axiom 3 in Russell’s order) fails. For if it is supposed that the total number of individuals in the universe is (say) 10, the number 11 would be the null-class. So the number 12 would also be the null-class. Thus it follows that 11 = 12. In other words, the successor of 10 = the successor of 11, but 10 \(\neq\) 11. This means that Peano’s axiom 4 (Russell’s axiom 3) would fail, because two different natural numbers would have the same successor. ‘However’, he argues, ‘this failure of the third [our fourth] axiom cannot arise, if the number of individuals in the world is not finite.’ (op. cit., 24) Thus at least in order to rescue Peano’s axiom 4, Russell had to rely on a metaphysical assumption such as his ‘axiom of infinity’.

\(^{12}\) Dummett (1991, 301.)
We can find in Frege’s Theorem in GLA a modest but consistent version of his logicism. Frege kept the context principle firmly in GLA. Although he gave the explicit definition of number with the extensions of concepts:

\[ \# F = \'X(F \approx X) \]

to solve the so-called ‘Julius Caesar problem’—the problem of how to determine the truth value of a sentence such as ‘\( \# F = \text{Julius Caesar} \)’—Frege derived the theorems of arithmetic using only the contextual definition, that is, Hume’s Principle:

\[ \# F = \# G \iff F \approx G. \]

Up till now Frege’s context principle—‘never to ask for the meaning of a word in isolation, but only in the proposition’ (Introduction of GLA)—has been too broadly interpreted. The principle has been promoted to the status of a general principle of philosophy of language or semantics. It is time to put Frege’s context principle back in the original context in which it arose. In GLA, Frege seems to have restricted his use of the context principle to giving a contextual definition of number. Although in the Introduction of GLA Frege set up the context principle as one of the three fundamental principles to keep to, his specific application of that principle seems to have been made in order to suggest how to avoid a physical or psychological view of number (§106 of GLA).

But Frege had to go further to adopt the explicit definition using a powerful notion, that is, the extension of concepts. Let us recapitulate. We can see in Frege’s Theorem a modest form of successful logicism. It uses only the weak notion of contextual definition of number in order to develop arithmetic. The way it adopts is too weak to answer the question of the essence of number, but it is free from inconsistency.

References

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