Essay Review


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1. Introduction

The book contains 30 essays by Boolos and is divided into three parts: ‘Studies on set theory and the nature of logic’ (essays 1 to 8), ‘Frege studies’ (9 to 21), and ‘Various logical studies and lighter papers’ (22 to 30). Each part is preceded by an introduction by Burgess putting the essays into context and highlighting Boolos’s work. In the afterword, Burgess gives a brief survey of the logic of provability, an area to which Boolos made substantial contributions but which, due to its technical character, is not represented in the book. The book has an index and its bibliography includes a complete list of Boolos’s works. Of the essays selected, only two (‘Reply to Charles Parsons’ Sets and Classes’ and ‘Gottlob Frege and the foundations of arithmetic’) are published here for the first time.

Most of the papers in the book are of philosophical interest; they deal, fully or partly, directly or indirectly, with problems in the philosophy of mathematics. The main areas discussed are Set Theory, Second-Order Logic and Plural Quantification, and Frege’s work on the Foundations of Arithmetic. I will center this review on these three topics.

2. Set theory

Zermelo-Fraenkel set theory (ZF) can be described as the theory of well-founded pure sets. According to a theorem of ZF which is equivalent, modulo the remaining axioms, to the axiom of foundation (which states that every non-empty set has a minimal element), sets are distributed in stages indexed by the ordinals, in such a way that a set belongs to a given stage if and only if all its members belong to some earlier stage. This ordinal-indexed sequence of stages is the cumulative hierarchy—cumulative, because each stage extends the preceding ones. In 1930, Zermelo included foundation among the axioms and described the cumulative hierarchy that they entail. This hierarchical distribution of sets was later seen as a justification of ZF, but it was largely unknown to philosophers until 1971, when Boolos published the first essay in this collection, ‘The iterative conception of set’.

The view according to which sets are distributed in stages as described is also called the ‘iterative conception’. Boolos argues that it is a natural conception and he explicitly articulates it in what he calls ‘stage theory’, a first-order axiomatic theory formulated in terms of sets and stages. Some of the axioms are about the ordering relation among stages, some describe at what stages sets are formed and what sets are formed at what stages. Finally, there is an induction schema to the effect that if all sets formed at any stage s have some property P provided all sets formed at stages earlier
than $s$ have that property, then all sets have property $P$, no matter at what stage they are formed.

Boolos shows how to derive from stage theory all axioms of ZF, except for extensionality and replacement (infinity follows because stage theory has an axiom asserting the existence of a stage without an immediate predecessor, and foundation is an easy consequence of the induction schema). As to extensionality, he argues that it is analytic, if anything is. On the other hand, to obtain replacement some principle beyond the iterative view of sets is needed. The same is true for the axiom of choice.

Essay 6, ‘Iteration again’, offers a simpler reformulation of stage theory, the main feature of which is the absence of any axiom that could be considered as inductive. The axiom of foundation is derived from this new version of the theory following an argument of Dana Scott,¹ which Boolos previously used in ‘The justification of mathematical induction’ (essay 24) to obtain induction for natural numbers from non-inductive principles about numbers and sets.

Given that neither replacement nor choice follow from the iterative conception, Boolos explores a new view about sets obtained by suitably modifying Frege’s theory of concepts. Frege had assumed that to each concept $F$ an object $\hat{F}$ corresponds (the extension of $F$) in such a way that two concepts have the same extension if and only if they are coextensive, i.e. if under both concepts the same objects fall. This is the content of (a restricted version of) *Grundgesetze*’s Axiom V:

\[(\text{Axiom V}) \quad \hat{F} = \hat{G} \text{ if and only if } F \text{ and } G \text{ are coextensive.}\]

As Russell showed, Axiom V is inconsistent. Boolos modifies axiom V as follows: call a concept $F$ ‘small’ if there is no one-one mapping assigning to every object some object falling under $F$. Call concepts $F$ and $G$ ‘similar’ if either none of them is small or else both are small and coextensive. Now assume that to each concept $F$ and object $*F$ is assigned ($F$’s ‘subtension’) so as to satisfy the so-called ‘new V’ principle:

\[(\text{New V}) \quad *F = *G \text{ if and only if } F \text{ and } G \text{ are similar.}\]

Boolos considers the theory FN (for Frege-von Neumann), which is standard axiomatic second-order logic supplemented with axiom new V. FN is consistent, being interpretable in second order Peano Arithmetic (PA²). Boolos shows how to do arithmetic in FN and he then proceeds to develop set theory as well, defining membership so that $x \in y$ if and only if $y$ is the subtension of a concept under which $x$ falls. A set is a subtension of some small concept. Boolos concentrates on pure sets (sets whose transitive closure contains only sets) and shows how to derive in FN all axioms of ZF relativized to them, except for infinity and power set, but including choice. In particular, FN yields those axioms that the iterative conception does not yield. ‘Perhaps’, Boolos says, ‘one may conclude that there are at least two thoughts “behind” set theory’ (p. 103).

It does not seem that Boolos gave to the thoughts behind FN the same weight as to the iterative conception, since in the later ‘Must we believe in set theory?’ (essay 8) he wrote that the ‘iterative conception is the only view of sets we have that is natural and, apparently, consistent’ (p. 127). In this essay, he expresses doubts about the acceptance of ZF, since it implies the existence of cardinals about whose existence he

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has some qualms; in particular, he objects to the existence of fixed points of the aleph function, that is of a cardinal $\kappa$ such that $\kappa = \aleph_\kappa$, which can be proven in ZF. The question arises because, as Boolos says, ZF is an interpreted theory whose variables range over all pure sets there are (p. 120); so, ‘if there are not as many as $\kappa$ things in existence, $\kappa$ does not exist, ... and set theory is not true’ (p. 121). It must be observed that one may consistently object to $\kappa$ while adhering to the iterative conception, because in order to prove $\kappa$’s existence the axiom of replacement is needed and Boolos denies that replacement follows from the iterative view of sets. Again he makes it clear that his reservations have nothing to do with nominalist scruples; he is not against the existence of abstract objects and he objects neither to $\aleph_0$ nor even to $2^{\aleph_0}$ (thus, as he remarks, his real target is rather the least fixed point of thebeth function instead of $\kappa$ (p. 123)).

In his edition of Cantor’s collected works, Zermelo commented on the acceptance of fixed points of the aleph function: ‘According to the “intuition” on which some philosophers and mathematicians want to found all mathematical knowledge, the existence of such numbers would be absolutely denied’. 2 This remark, however, does not quite apply to Boolos. For, on the one hand, Boolos does not absolutely deny the existence of $\kappa$, but only expresses doubts about it; on the other hand, the viewpoint from which he objects to the existence of $\kappa$ is hardly the same as that intimated by Zermelo. According to Boolos (p. 130, italics in the original),

What we are contemplating here ... is ... the rejection of ... the claim to be a body of knowledge on the part of a portion of set theory that treats of objects far removed from ordinary experience, the rest of physical science, the rest of mathematics, and the rest of a certain more ‘concrete’ part of set theory.

3. Second-order logic and plural quantification

Second-order logic has been consistently attacked by Quine. He has portrayed it as being set theory in disguise (‘set theory in sheep’s clothing’), since quantifying in predicate positions amounts to quantifying over sets. Accordingly, he has urged to avoid second-order quantification altogether and quantify explicitly over sets by introducing set variables, thus writing ‘$x \in \alpha$’ instead of ‘$Fx$’, and quantifying on ‘$\alpha$’ if one wants to quantify on ‘$F$’. Quine also has objected to the allegedly heavy existential commitments of second-order logic, which are due to its hidden set-theoretical nature.

In ‘On second-order logic’ (essay 3), Boolos argues against Quine’s view of second-order logic. He examines Quine’s argument against quantifying on predicate positions and finds it unsound. He also rejects Quine’s suggestion of reformulating second-order logic set-theoretically, mainly because the set-theoretical rewriting of second-order formulas may result in the loss of validity or implication (thus ‘$\exists F \forall x Fx$’ is valid, but ‘$\exists \alpha \forall x x \in \alpha$’ is not). Against the charge that second-order logic involves strong assumptions about sets, Boolos remarks that no second-order validity can assert the existence of any sets outright, but only of some subsets of the domain of interpretation. He concludes that ‘the null set is the only set to whose existence second-order logic can be said to be committed’ (p. 48). Logic is committed to it, Boolos says, because the sentence ‘$\exists X \forall x \neg Xx$’ may be taken to assert the existence of the empty set, independently of the domain of interpretation.

In essay 3, Boolos requires that the domain of any interpretation of a second-order language be a set, so that we cannot use a second order language to talk about the full set-theoretical universe. Thus a sentence such as:

\[ \exists X \forall x (Xx \leftrightarrow \neg x \in x) \]

is logically valid, but it cannot be read as asserting that there is a set whose members are all the non-selfmembered sets.

Later, however, Boolos used second-order formulas even when the first-order variables did not range over a set. In particular, he viewed as legitimate to use a second-order language to talk about all sets. But then some explanation is needed about how to understand formulas like (*). What does the variable \( X \) range over? If it ranges over sets, then (*) is no longer valid. Must it range over so-called proper classes, then? Boolos objects to proper classes, since set theory, he says, ‘is supposed to be a theory about all set-like objects’ (p. 66, italics in the original).

In ‘To be is to Be a Value of a Variable (or to Be some Values of Some Variables)’ (article 4) and ‘Nominalist Platonism’ (article 5), Boolos proposes understanding second-order quantification as plural quantification over the same individuals which are the values of the first-order variables—not as ordinary, singular quantification over sets or classes of those individuals.

Thus, ‘\( \exists X \)’ is to be read as ‘there are some objects such that ...’. This allows for a natural rendering of second-order set-theoretical formulas. On this reading, (*) does not say that there is a set or a class containing all and only those sets which are not members of themselves; what it says is simply that there are some sets such that any set is one of them if and only if it is not a member of itself.

In general, in order to understand a second-order formula (of an interpreted language) we are to translate it into our mother tongue by using the plural forms that are available in it. The intelligibility of plural constructions and of the sentences involving them is taken for granted, but this assumption does not presuppose that we can give a set-theoretical semantic theory for them (p. 70):

In view of the work of Tarski it should not automatically be expected that we can give an adequate semantics for English—whatever that might be—in English. Nothing whatever about the intelligibility of those sentences would follow from the fact that a systematic semantics for them cannot be given in set theory. After all, the semantics of the language of ZF itself cannot be given in ZF.

An important feature of the plural reading of second-order quantification is that it allows us to express the first-order separation and replacement schemas of ZF with single sentences, without having to assume the existence of set-like objects other than sets as values of our second-order variables. This is of considerable value, Boolos argues, because (65, italics in the original):

Whatever our reasons for adopting Zermelo-Fraenkel set theory in its usual formulation may be, we accept this theory because we accept a stronger theory consisting of a finite number of principles, among them some for whose complete expression second-order formulas are required. We ought to be able to formulate a theory that reflects our beliefs.

A simple plural reading of the separation and replacement axiom can be found in David Lewis’s Parts of classes: ‘Given a set \( x \), and given some things, there is a set of all and only those of the given things that are members of \( x \).’

A problem arises when trying to translate into natural language binary and, in general, polyadic second-order quantification. About how to solve this problem, which does not affect set theory, because of the definability of a pairing function, Boolos makes some suggestions in ‘1879’ (essay 15, pp. 243–244).

4. Frege

In Grundlagen de Arithmetik, Frege provided a definition of natural number and sketched a proof, to be carried out in Grundgesetze, that the numbers so defined satisfy what we now know as the Peano Axioms. Let us say that concepts $F$ and $G$ are ‘equinumerous’ (in symbols, $F \approx G$) if there is a one-to-one correspondence between the objects falling under $F$ and the objects falling under $G$. To each concept $F$, Frege wants to assign an object $\# F$, the number of $F$, so as to satisfy Hume’s principle (HP): 

$$\# F = \# G \iff F \approx G.$$

Frege defines $\# F$ as the extension of the second level concept ‘concept equinumerous with concept $F$’. To see that this definition does yield Hume’s principle, we just need to observe that both $F \approx G$ and $\# F = \# G$ are equivalent to the condition $(\forall H)(H \approx F \leftrightarrow H \approx G)$; the former, because equinumerosity is an equivalence relation, the latter because of the rule governing extensions, namely, Frege’s Axiom V. Once secured the general notion of number, Frege defines the number zero, the relation of immediate precedence between numbers and finally he defines a natural number to be either 0 or else any number to which 0 bears the ancestral of the relation of immediate precedence. This amounts to defining a natural number as a number satisfying the principle of induction.

Since numbers are construed as extensions and Frege’s basic principle about extensions (Axiom V) is contradictory, Frege’s definition of natural numbers can be taken to be a failure. However, in ‘Frege’s conception of numbers as objects’ (1984), Crispin Wright showed how the fundamental properties of natural numbers can be derived in the second-order logic of Begriffsschrift from HP without any use of Axiom V. He also showed that from HP the usual derivation of Russell paradox cannot be carried out. Indeed, as John Burgess and, with more detail, Allen Hazen indicated in their reviews of Wright’s book, HP is in fact consistent.

In the second part of the book under review, Boolos dwells on several aspects of Frege’s work on logic, mainly as it relates to numbers, both from a technical and from a philosophical point of view. Boolos shows that what he calls ‘Frege’s arithmetic’ (FA), viz. second-order logic with HP as a further axiom, is consistent relative to second-order Peano arithmetic (PA²). The idea is to define the map $\#$ in PA² in such a way that every infinite set is mapped to zero, while a finite set of $n$ elements is mapped to $n + 1$. With respect to the derivation of the Peano Axioms from HP (that is, in FA), Boolos shows that the very proof outlined by Frege in Grundlagen, which he reconstructs (pp. 217–219), does not appeal to Axiom V, so that Frege himself can be said to have had a derivation of PA² in FA. This can be seen as a vindication of Frege’s work on the foundations of arithmetic.

The vindication, however, is only partial, since one of Frege’s aims was to show that arithmetic is part of logic, and Boolos argues at different places that HP is not a

logical truth, not even an analytical one, in any reasonable sense of analytic. One reason against viewing HP as a logical principle or as an analytic truth is that HP implies the existence of infinitely many objects. Another reason concerns the status of Fregean contextual definitions (like Axiom V, new V, or HP), each based on a particular equivalence relation. Following Burgess, any such definition is a principle according to which objects are assigned to second-order entities (say concepts) in such a way that the objects assigned to concepts $F$ and $G$ are the same if and only if $F$ and $G$ are equivalent (p. 137). As Axiom V makes clear, there are inconsistent contextual definitions. So, why should we bestow on HP the title of a logical or an analytical truth if some principles of its form are contradictory?

But even allowing that we are able to discern which contextual definitions are consistent, it turns out that there are inconsistent pairs of independently consistent contextual definitions. Let us say with Boolos (pp. 214–5) that concepts $F$ and $G$ ‘differ evenly’ if and only if their symmetric difference (i.e. the set of all objects falling either under $F$ but not under $G$, or under $G$ but not under $F$) is finite and even. Such a relation is an equivalence relation among concepts. Now assign to each concept $F$ its parity so that the following ‘parity principle’ holds: the parity of $F$ is the same as the parity of $G$ if and only if $F$ and $G$ differ evenly. In order to see that that the parity principle is consistent, consider any finite domain $D$ containing the numbers 0 and 1 and let the parity of any subset of $D$ be 0 or 1 according as it contains an even number or an odd number of elements.

But the parity principle cannot be satisfied in any infinite domain. This is so, because if $D$ is an infinite set of cardinality $\kappa$ and $F \subseteq D$, only $\kappa$ many subsets of $D$ differ evenly from $F$. It follows that there must be $2^\kappa$ equivalence classes, and, hence, no assignment of parities is possible in such a way that different members of $D$ are assigned to non-equivalent sets.

Thus both the parity principle and Hume’s principle are consistent, but each is inconsistent with the other. ‘Which’, Boolos asks rhetorically, ‘is the logical truth?’ (p. 215).

The status of contextual definitions is discussed at different places in the book, among them in the stimulating essay 14 (‘Whence the contradiction?’), where Boolos argues, against Dummett, that the origin of Frege’s contradiction is to be found in Axiom V, rather than in the use of second-order quantification. The interest of such a discussion is enhanced by Terence Parsons’s proof (sketched on p. 230) that the first-order fragment of Grundgesetze is consistent. The existence of this proof, Boolos argues, is not enough to show that Dummett’s diagnosis is right, since the fragment is too weak for the proper development of arithmetic (p. 235):

only if the first order fragment had been strong enough to yield arithmetic or an interesting portion of it, would it be tempting to ascribe the inconsistency to the second-order quantifiers. For example, if some constructive or predicative second-order version of Frege’s system could be defined and shown to be consistent and adequate for arithmetic, then Dummett’s claim would acquire a force it now lacks.

This was written in 1993. In 1996, Richard Heck showed that the predicative fragment of Frege’s Grundgesetze der Arithmetik is indeed consistent. As to its arithmetical strength, he also showed that Robinson’s theory $Q$ is relatively interpretable in it; but $Q$ can be hardly regarded as adequate for arithmetic.

I have described the main themes in Boolos’s essays, although I have not attempted to convey the degree to which they are intertwined in the book, which has a much higher unity than either this review or the reading of the table of contents would let one suspect. It has a larger scope as well, touching upon other aspects of logic, be it from a technical, a historical or a philosophical point of view. This book can be of use to people with a variety of interests, to philosophers interested in the philosophy of logic and mathematics, and to logicians and mathematicians concerned with foundations. In the editorial preface, Burgess and Jeffrey tell us that the book was designed by Boolos himself shortly before his untimely death in 1996. We must be grateful to them for having brought Boolos’s project to completion, allowing us to better recognize the range and subtlety of this highly stimulating work.